



Instituto de Astrofísica de Andalucía,
Consejo Superior de Investigaciones Científicas

Departamento de Física Teórica y del Cosmos,
Universidad de Granada

**Group-theoretical quantization
of non-linear systems and dissipative systems**

Tesis Doctoral

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Diciembre 2010

Editor: Editorial de la Universidad de Granada
Autor: Francisco F. López Ruiz
D.L.: GR 1378-2011
ISBN: 978-84-694-0927-5

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CERTIFICAN:

Que la presente Memoria "**Group-theoretical quantization of non-linear systems and dissipative systems**" ha sido realizada bajo nuestra dirección por **Francisco Felipe López Ruiz**, en el Instituto de Astrofísica de Andalucía (CSIC) y a lo largo de una estancia en el **Imperial College** de Londres, y constituye su Tesis para optar al grado de Doctor.

Y para que así conste, en cumplimiento de la legislación vigente, presentamos ante la Comisión de Doctorado de la Universidad de Granada la referida Tesis.

Granada, a 15 de Diciembre de 2010.

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"Group-theoretical quantization of non-linear systems and dissipative systems"

El presente trabajo ha sido realizado en el Instituto de Astrofísica de Andalucía - CSIC, bajo la dirección de los Doctores Víctor Aldaya Valverde, Manuel Calixto Molina y Julio Guerrero García.

En Granada, a 15 de diciembre de 2010

Francisco F. López Ruiz

Agradecimientos

Quisiera expresar aquí, aunque sea torpemente, la gratitud que siento hacia todos los que me han ayudado, de una manera u otra, a concluir este trabajo.

En primer lugar, quiero agradecer a mi director de Tesis, Víctor Aldaya, toda la ayuda y formación recibidas, su dedicación inigualable, su paciencia. Y su compañía y amistad durante esta etapa.

Quiero agradecer también a Julio Guerrero y Manuel Calixto, mis codirectores, todo lo que he aprendido con ellos. Junto a Víctor han hecho que sienta que pertenezco a un grupo, no sólo científico sino humano, girando en torno a física, grupos, tés, pizarras, skype y Mathematica.

Gracias a José Luis Jaramillo, por los ánimos y por el ejemplo. Y a Carlos Barceló.

Mi gratitud a Hugh Jones por su cálida hospitalidad durante mi estancia en Imperial College de Londres.

Agradezco también al CSIC la concesión de una beca I3P.

La realización de una tesis no sólo precisa de unos mentores como los que he tenido, sino también de un equilibrio emocional que sólo se consigue con el entorno adecuado. Me gustaría dar las gracias a aquellos que han contribuido a mantener mi ánimo y mi cordura, dentro de lo que cabe, sirviéndome de un apoyo sin el que hubiera sido imposible llegar hasta aquí.

A mis padres, Paco y María Dolores, por todo el apoyo y el cariño que recibo de ellos. Son los mejores padres que se pueda tener.

Y a mi hermano Alejandro, por estar ahí para poder recurrir él, tengo mucha suerte; por ser además un ejemplo y por sacarme para coger la bici cuando mi pereza me retenía.

Gracias a mi tío Enrique, por ser un estímulo cada vez que hablamos de cualquier cosa y de Física en particular. Tenemos algo pendiente.

A mis amigos del instituto: Javi, Julio, Pedro, Ismael, Jose...Y a Pablo. Nos vemos mucho menos de lo que quisiera, pero os llevo siempre conmigo. Os quiero.

También quiero darles las gracias a aquellos que han hecho del IAA mi casa. Al Rabino, por las rabinoaventuras. Junto a Andrés y a mi hermano, me ha hecho recordar cuánto me gusta la bicicleta. A los que han sido mis compañeros de despacho en momentos clave: Mar al llegar; Martiña, por hacerme pasar buenos ratos teniéndola al otro lado de la mesa; Vicent, ¡hemos hecho el viaje juntos!; y a todos aquellos con los que también conviví en el zulo y más allá. Sé que soy un poco huraño, y a pesar de eso me he sentido muy querido.

Y sobre todo, a Isa, por todo el amor del mundo. A ella dedico este trabajo.

Para Isa

Resumen

La idea central que guía este trabajo es investigar modelos físicos que no son bien entendidos desde el punto de vista de la Teoría Cuántica estándar. Por un lado, sistemas no lineales y en particular, entre ellos, aquellos cuya no linealidad proviene del hecho de tener variedades de soluciones no planas, es decir, con topología no trivial, incluso en teorías de campos. Por otro lado, sistemas físicos que, aunque con variedad de soluciones plana y dinámica lineal, incorporan cierto tipo de comportamiento disipativo.

Como ejemplos paradigmáticos de sistemas no lineales, con variedad de soluciones no plana, nos fijaremos en modelos sigma no lineales (NLSM), tanto en el caso de número de grados de libertad finito como infinito (teorías de campos). Esto nos permitirá embarcarnos en una tarea más ambiciosa: el estudio de un mecanismo de generación de masa para bosones vectoriales intermedios en teorías de Yang-Mills, es decir, el mecanismo de Stueckelberg no abeliano, que se basa parcialmente en un NLSM.

El papel principal en nuestro estudio de sistemas disipativos lo jugará el oscilador armónico amortiguado y los modelos cuánticos que lo describen: el modelo de Caldirola-Kanai y sistema dual de Bateman. Sin embargo, la comprensión de estos modelos será útil más allá, y sugerirá interesantes relaciones entre el conjunto de todos los sistemas lineales de una partícula.

El tema recurrente de esta tesis, que guiará muchas líneas de razonamiento, será el concepto de simetría de un sistema físico como el grupo de transformaciones que dejan invariantes objetos que caracterizan dicho sistema, y su profunda conexión con la formulación del correspondiente sistema cuántico. Es más, la simetría adquirirá el estatus de piedra angular en la definición misma del sistema físico y consideraremos que, en cierto sentido, un sistema físico *fundamental* es, él mismo, el conjunto de sus simetrías.

El proceso llamado “cuantización” está íntimamente relacionado con ciertas propiedades de simetría. La Cuantización Canónica se basa, de hecho, en asumir que la simetría de la variedad de soluciones es la del grupo de Heisenberg-Weyl, correspondiente al caso plano, y representarlo unitaria e irreduciblemente. Esto viene sugerido, implícitamente, por el teorema de Darboux, que afirma que siempre es posible encontrar un conjunto de coordenadas canónicas *localmente*. Además, los teoremas de “no-go” de Gronwald y van Hove establecen que es imposible cuantizar de manera consistente más allá de los polinomios de segundo grado en las coordenadas canónicas básicas si nos apoyamos en la prescripción de la cuantización canónica.

Para llegar a una teoría cuántica consistente en los sistemas físicos que se plantean en la memoria, recurriremos a la Cuantización Sobre Grupos (GAQ en inglés). Se trata de un algoritmo para obtener representaciones unitarias e irreducibles de grupos de simetría, clasificadas según sus extensiones centrales por $U(1)$. Al aplicar este algoritmo al grupo

de simetría básico de un sistema, es posible obtener la teoría cuántica directamente, sin las obstrucciones propias de la Cuantización Canónica.

Nos acercaremos de tres formas a la cuestión de encontrar una estructura de grupo en la que basar la GAQ:

1. La primera consistirá en seleccionar (y cerrar) una subálgebra de Poisson de funciones de las variables canónicas básicas y de la hamiltoniana. La exponenciación de ese grupo permite aplicar GAQ y cuantizar. Sólo algunas funciones serán representadas en la teoría cuántica correspondiente.
2. La segunda será buscar un sustituto del grupo básico de Heisenberg-Weyl de la Cuantización Canónica, basándonos en un análisis de la simetría de la variedad de soluciones del sistema concreto. Sólo algunos sistemas peculiares admiten este tratamiento.
3. Hay otra posibilidad puramente algebraica, consistente en construir un modelo físico directamente a partir de un álgebra de Lie abstracta que codifique lo fundamental del sistema físico.

El plan de la tesis empieza en el Capítulo 2 por una introducción detallada a las técnicas de Cuantización Sobre Grupos. El Capítulo 3 se dedica al estudio de varios modelos con un número finito de grados de libertad, primero lineales (partícula en campos electromagnético y gravitatorio) y luego no lineales: partícula moviéndose en la variedad de $SU(2)$ vista como modelo sigma no lineal mecánico, potenciales de Pöschl-Teller y Morse y el movimiento de una partícula en una esfera S^2 , donde las simetrías de la variedad de soluciones son identificadas, así como un hamiltoniano que, si bien no cierra álgebra con los operadores básicos seleccionados, respeta la representación dada por los mismos. No es necesario un tratamiento explícito de las ligaduras, el grupo las incorpora de manera natural, y el espectro de energía se obtiene sin términos inesperados en la curvatura. El análisis de este último modelo será relevante para llegar al caso de sistemas tipo modelo de Stueckelberg, que proporcionan una teoría de Yang-Mills masiva.

El Capítulo 4 se dedica a reconsiderar, desde primeros principios de la teoría cuántica, el modelo de Stueckelberg, que podría constituir una alternativa al mecanismo de Higgs-Kibble de generación de masa en el Modelo Estándar de Física de Partículas. Con las técnicas de Cuantización Canónica, dicho modelo resulta ser o bien no unitario o bien no renormalizable. En este capítulo se encuentra el grupo relevante para la cuantización del sistema, y se proporciona una representación unitaria e irreducible del mismo. Además, puede encontrarse de manera inambigua un hamiltoniano que respeta la representación y cuya versión clásica proporciona las ecuaciones de movimiento.

No está dentro de los objetivos de la tesis tratar cálculos convencionales de teoría de perturbaciones en teoría cuántica de campos, sino establecer los fundamentos en los que éstos podrían apoyarse rigurosamente. En el Capítulo 5 planteamos una posible aproximación a estos cálculos para el modelo sigma no lineal invariante $O(N)$ en un régimen perturbativo diferente del usual, tratando de sacar partido de relaciones de conmutación

no canónicas básicas adaptadas a la simetría $O(N)$ del sistema, que también dan cuenta de la geometría y topología no triviales de la variedad en la que toman valores los campos sigma.

Con una orientación diferente, el propósito del Capítulo 6 es arrojar algo de luz en el estudio de la disipación cuántica, con la guía de la simetría. Se generaliza al régimen cuántico la transformada de Arnold de la mecánica clásica, que relaciona todos los sistemas cuya ecuación clásica es una ecuación diferencial de segundo orden no homogénea de coeficientes arbitrarios con el sistema de la partícula libre. Con la Transformada de Arnold Cuántica (QAT), se importan operadores básicos de la partícula libre, así como el grupo de Schrödinger, que puede a su vez realizarse sobre el oscilador amortiguado de Caldirola-Kanai. Además, la QAT es muy útil para hacer cálculos rápidamente en los que se obtienen funciones de onda, el propagador cuántico o el operador evolución de manera exacta.

Con la ayuda de la QAT entre la partícula libre y el oscilador armónico simple, se construye unas series de soluciones de la partícula libre que constituyen bases discretas del espacio de Hilbert. Estas soluciones son normalizables, espacialmente localizadas, y se caracterizan por ser autoestados de un cierto operador número importado del oscilador armónico. Se construyen a su vez los estados coherentes asociados. Estos estados se construyen también en dimensiones espaciales mayores y dan lugar a la versión mecano-cuántica de los estados de Hermite-Gauss y Laguerre-Gauss de la óptica ondulatoria paraxial.

Las traslaciones temporales en el sistema no libre no pertenecen, en general, al grupo de Schrödinger importado desde la partícula libre. Esto es esperable, ya que las ecuaciones clásicas de movimiento incluyen un término de fricción de manera que la energía del sistema no se conserva. La siguiente pregunta surge inmediatamente: ¿existe algún grupo de simetría de dimensión finita que contenga las traslaciones temporales y, al menos, los operadores básicos? La respuesta es “sí”. Para que esa simetría actúe adecuadamente, es necesario modificar el sistema físico para que contenga, además, un nuevo grado de libertad correspondiente a una nueva partícula con interesantes propiedades. Esto podría entenderse como una versión muy simple del Principio de gauge, en el que se impone una simetría mayor que la que tiene originalmente el sistema libre. De hecho, el nuevo sistema con dos grados de libertad es el sistema dual de Bateman. Aprovechando la estrategia basada en simetrías, iremos un poco más lejos y proporcionaremos una ley de grupo correspondiente a las simetrías del sistema dual.

A la luz de esta ley de grupo, damos un análisis de la cuantización del sistema de Bateman que hemos encontrado. En particular, mostramos que es posible encontrar una ecuación de Schrödinger de primer orden, de la que se obtienen las funciones de onda y el espectro de energía, así como la ecuación más convencional de segundo orden. Ilustramos también cómo el sistema de Caldirola-Kanai puede reobtenerse mediante un proceso de ligadura.

Esta memoria se basa fuertemente en los siguientes trabajos publicados y en preparación:

- V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *Quantum Integrability of the Dynamics on a Group Manifold*, J. Nonlinear Math. Phys. **15**, 1 (2008).

- M. Calixto, V. Aldaya, F.F. López-Ruiz and E. Sánchez-Sastre, *Coupling Nonlinear Sigma-Matter to Yang-Mills Fields: Symmetry Breaking Patterns*, J. Nonlinear Math. Phys. **15**, 91 (2008).
- V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *Group-quantization of non-linear sigma models: particle on \mathbb{S}^2 revisited*, Rep. Math. Phys. **64**, 49-58 (2009).
- J. Guerrero, F.F. López-Ruiz, M. Calixto and V. Aldaya, *On the geometry of the phase spaces of some $SO(2,1)$ invariant systems*, Rep. Math. Phys. **64**, 329-340, (2009).
- V. Aldaya, M. Calixto and F.F. López-Ruiz, *A Quantizable Model of Massive Gauge Vector Bosons without Higgs*, Mod. Phys. Lett. A **24**, No. 34, 2731-2740, (2009).
- V. Aldaya, M. Calixto and F.F. López-Ruiz, *Non-Canonical Perturbation Theory of Non-Linear Sigma Fields*, accepted for publication on Oct 18, 2010, in Mod. Phys. Lett. A, arXiv:1007.2820.
- V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz, *The quantum Arnold transformation*, accepted for publication in J. Phys. A; arXiv:1010.5521.
- V. Aldaya, M. Calixto and F. F. López-Ruiz, *Symmetry group for massive Yang-Mills theories*, in preparation.
- V. Aldaya, M. Calixto, J. Guerrero and F. F. López-Ruiz, *Symmetries of Non-Linear Systems: Group Approach to their Quantization*, in preparation.
- J. Guerrero, F.F. López-Ruiz, V. Aldaya and F. Cossío, *Discrete basis of localized quantum states for the free particle*; arXiv:1010.5525.
- V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz, *A symmetry trip from Caldirola to Bateman damped systems*, in preparation.

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Chapter 1

Introduction

The central idea guiding this work is to investigate physical models which are not well understood from the point of view of the standard Quantum Theory. On the one hand, non-linear systems and, in particular, those whose non-linearity comes from the fact that their classical solution manifolds are not flat, that is, those possessing non-trivial topology, even in field theories. On the other hand, physical systems that, although with a flat solution manifold and linear dynamics, incorporate some kind of dissipative behavior.

As paradigmatic non-linear systems, with non-flat solution manifold, we will focus on non-linear sigma models (NLSM), both in the case of finite degrees of freedom (the mechanical case), and in that of infinite degrees of freedom (field theories). This will allow us to embark on a more ambitious program: the study of a mass-generating mechanism for the intermediate vector bosons in Yang-Mills theories, namely, the non-Abelian Stueckelberg mechanism, which is partly based on a NLSM.

The main character of our study of dissipative systems will be played by the damped harmonic oscillator and the quantum models describing it: the Caldirola-Kanai model and the Bateman's dual system. However, the insight gained in this analysis finds applicability beyond this class of linear models and interesting relationships between all linear systems will be established.

The recurring theme of this doctoral thesis, driving many of the lines of reasoning, will be the concept of symmetry of a physical system as the group of transformations leaving invariant the objects characterizing such system, and its deep connection with the formulation of the corresponding quantized system. Furthermore, symmetry will acquire the status of cornerstone in the very definition of a physical system and we will consider that, in some sense, a *fundamental* physical system itself *is* the set of its symmetries.

The idea of quantization, that is, the process of obtaining a quantum theory out of a given classical model, is in fact rooted in certain assumed symmetry properties. The quantization method known as Canonical Quantization, which relies on Hamiltonian mechanics and in which Quantum Mechanics finds its original foundations at the beginning of the 20th century, could be summarized as follows [1]: consider Cartesian coordinates q_j of configuration space of a given classical system and their corresponding conjugate momenta p_k ; then, the operators \hat{X}_j and \hat{P}_k which represent these observables must satisfy the commutation relations $[\hat{X}_j, \hat{P}_k] = i\hbar\delta_{jk}$ (the rest of the commutators are zero). This rule can be easily generalized to the case of field theories, and has been considered appropriate to deal with even non-linear systems. Implicitly, this prescription assumes that the symmetry of the classical solution manifold of the physical system to be quantized

corresponds *globally* to the Heisenberg-Weyl group in the corresponding (even infinite) dimension. The replacement prescription of Canonical Quantization to get quantum operators amounts (through the Stone-von Neumann theorem) to obtain one of the unitarily equivalent, weakly continuous representations of this group and the corresponding Lie algebra representations (“position” and “momentum” representations for instance, to be intuitive) in terms of self-adjoint operators. However, the Darboux theorem of Classical Mechanics, which states that choosing canonical coordinates in the solution manifold is always possible *locally*, might turn out to be misleading at this point, suggesting taking this “tangent-space approximation” for real. This would imply to take the symmetry under the Heisenberg-Weyl group itself and therefore a proper representation of this group, corresponding to Canonical Quantization, for granted. However, this is rarely true in non-linear systems.

Moreover, Canonical Quantization turns out to be ambiguous in general when we try to obtain the quantized operators corresponding to arbitrary functions of q_j and p_k belonging to the general Poisson algebra. The Heisenberg-Weyl representation is unique up to unitary equivalence; it determines a unique projective representation of the symplectic group from general principles of representation theory. The so-called “no-go” theorems by Groenwald and van Hove [2, 3, 4] state that there is no way to extend the representation of the universal covering group of the symplectic group to include any nonquadratic polynomial: we cannot get beyond quantizing quadratic polynomials. Usually, a prescription of normal ordering for operators is given ad hoc to avoid ambiguities. Even in the path integral approximation to quantization, this ambiguity is “hidden” in the necessity of choosing a particular prescription for evaluating position differences (corresponding to velocities in the continuum limit) at each point of the time slicing of the paths [5]. It is clear that a neater guide in the whole program would be desirable.

A step ahead in looking for an improved strategy of quantization of non-linear systems is to look for subalgebras of the entire Poisson algebra which are, in some sense, better adapted to the actual global properties of the solution manifold than the Heisenberg-Weyl one, and including, at least in their enveloping algebra, those functions physically relevant, mainly the Hamiltonian. Projective representations (the relevant ones in Quantum Mechanics) of this subalgebras would then constitute a quantization of the system. This possibility will be exploited in this dissertation.

Generalizing this approach even further, we can try to identify the actual symmetry group of the classical solution manifold, whichever it might be, and to find a unitary and irreducible representation of this group. The basic, self-adjoint operators providing a proper quantization would then be considered to be as those infinitesimal generators in this representation. We can push forward and consider this a *natural* and *fundamental* prescription, which generalizes canonical quantization.

According to this approach, it will be possible to represent in the emerging quantum theory only constants of motion, i.e. conserved quantities well defined in the classical solution manifold. This suggests to slightly rethink the case of linear systems. There, the basic symmetry happens to be the Heisenberg-Weyl group and, in principle, Canonical Quantization should work representing up to quadratic functions. However, it is important to note that one should make sure that basic functions q_j and p_k , to be promoted to basic quantum operators, are Noether invariants associated with this basic symmetry. As

a consequence, \hat{X}_j and \hat{P}_k will be symmetry generators. This observation is crucial in the treatment of dissipative systems that will follow.

It could be argued that such an emphasis in finding proper basic operators to build the quantum theory is not essential, considering that one should only care about having a self-adjoint Hamiltonian operator to describe the system. This is far from being true. The Hamiltonian operator provides the time evolution of the system and hence the dynamics. But the kinematics is encoded in the basic operators and their commutation relations. Then, the basic symmetry of the system is vital to determine the observable self-adjoint operators, the physically realizable pure states or the superselection rules. The Hamiltonian by itself does not provide this information.

In the task of achieving the quantum description of some fundamental physical systems, and in the spirit of the improvements outlined above, we will appeal to the features of a group-theoretical approach which is being developed over the last decades, a Group Approach to Quantization (GAQ) [7]. It attempts to contribute to the big effort that had been devoted to place Quantum Mechanics in a similar geometrical status to that of Classical Mechanics or, even, General Relativity, in that which was known as Geometric Quantization (GQ) [8, 9, 10, 11], and is somewhat related to the methods developed by Isham [103] or Klauder [12], among others.

The main ingredient in GAQ is the group structure taken to the ultimate consequences, that is to say, symmetry is intended to contribute to Physics as a building block rather than a practical tool for finding additional solutions to partially solved (symmetrical) problems. Even more, this approach attempts to describe a quantum physical system from the group manifold itself and its canonical structures, aiming at reducing the problem of establishing the physical postulates to that of choosing specific groups. In addition, it should be considered as a method for describing directly the quantum dynamics since the intermediate step of solving the classical equation of motion is not required. In fact, the quantum nature of a given system can be associated with the actual (compact) topology of (part of) the addressing symmetry group, namely a central extension of a certain basic group by $U(1)$, whereas the classical limit is obtained by simply taking a local version (in the sense of taking a local chart) of this symmetry (opening the multiplicative $U(1)$ central subgroup to the additive real line \mathbb{R}).

From a technical point of view, this method also represents significant advantages. In particular, the biggest obstruction found by Geometric Quantization in dealing with non-linear systems, that of achieving the complete reduction of the geometric representation (polarization), can now be much better addressed on the grounds of the algebraic group structure. This is mainly due to the existence of two mutually commuting (left- and right-) actions, so that the infinitesimal generator of one of them can be used to construct the Poisson (classical) algebra representation (pre-quantization in the sense of Geometric Quantization), whereas the other can be employed to reduce completely the representation (true quantization).

Although the requirement of the additional structure of Lie group might be seen as a drawback, it should be remarked that after all, the Lie algebra structure is one of the few bricks shared by all quantization methods, which look for unitary and irreducible representations of a given Lie (Poisson) algebra somehow characterizing a physical system.

We will consider three main attitudes, already outlined above, to confront the quantization of a given system using a symmetry group structure, summarized in the following points:

1. The first one will consist in selecting a Poisson subalgebra, built out of functions of the canonical variables and the Hamiltonian function. This will imply a proper deformation of these original functions which, in general, do not close an algebra of observables and hence can not be exponentiated to a Lie group when considered as an abstract structure. The obvious advantage is that this strategy will permit us to use the machinery of the GAQ. Only some functions can then be represented, discarding the quantized version of those original classical functions on the solution manifold presenting possible ordering problems. This method was already applied in [6] to study the dynamics of the modified Pöschl–Teller potential and here will be further developed.
2. The second approach will be to look for a substitute of the basic Heisenberg-Weyl group postulated by Canonical Quantization. On the representation space of this substitute group, a Hamiltonian operator might be found, which respects the quantum representation space. This will not be possible in general and only special systems allow this treatment.
3. There is yet another possibility, a pure algebraic one, consisting in building a physical model right through the construction of an abstract Lie algebra which encodes the fundamentals of the particular physical system. This matches the frame of mind of GAQ, and intuition on how the physics is encoded in these structures will then be the main guide.

The plan of this thesis begins in Chapter 2 by a detailed introduction to the techniques of the Group Approach to Quantization. General considerations about quantization are made in Section 2.1. In Section 2.2 the notion of semi-invariance of a classical system is introduced and the necessity of considering central extensions of symmetry groups to describe a quantum system is also established. The precise formalism is reviewed in Section 2.3 and, finally, in Section 2.4 the simple examples of the free Galilean particle and the particle moving on a circle are worked out.

Chapter 3 is devoted to the study of some mechanical models with a finite number of degrees of freedom. We start by illustrating the third above-mentioned strategy to build a physical system revisiting the minimal coupling principle for a particle moving on an external field, either electromagnetic or gravitational (Section 3.1). However, given a classical physical system we need a more flexible plan. Former point 1 will be exemplified in Section 3.2 in the case of a relativistic free particle, regarded as a non-linear system. This will be useful in the next two sections, where we face true non-linear systems: a particle moving in the $SU(2)$ manifold (Section 3.3) and in Pöschl-Teller and Morse potentials (Section 3.4). Then we move to Section 3.5: the description of a particle moving in a two-sphere S^2 . This system can be regarded as the classical analogue of a special kind of non-linear sigma model, a “partial trace” NLSM on a group orbit. The analysis of this simple, constrained system will be very relevant in going ahead to more complex

models. We will identify the symmetries in the solution manifold so that we get a candidate to replace the Heisenberg-Weyl algebra of canonical commutation relations and we will demonstrate the power of this approach. In particular, the possibility of identifying the correct Hamiltonian operator, respecting the representation space of the quantum system, even though it does not close a Lie algebra with just the basic ones. No explicit constraint treatment is required nor ordering ambiguities do appear. Moreover, the energy spectrum is recovered without extra terms in the curvature of the sphere apart from those coming from the Laplace-Beltrami operator. The fact that the partial trace NLSM allows this treatment could be considered as an indicator of the special characteristic of this system. It will be shown later in the next chapter that the field-theoretic version of this model is the basis of a Stueckelberg mechanism for massive Yang-Mills theories, in turn a building block for constructing an alternative to the Higgs-Kibble mechanism.

Chapter 4 is devoted to a reconsideration, from first principles of quantum theory, of genuinely massive Yang-Mills models, which would give an alternative explanation of the masses of the intermediate vector bosons W^\pm and Z^0 . We refer to what is generically known in the literature as “Stueckelberg models”.

The practical interest in massive Yang-Mills theories is fairly obvious. The theory nowadays broadly accepted to describe electroweak interactions, the Standard Model of Particle Physics, is formulated as a $SU(2)_L \otimes U(1)_Y$ gauge theory. To find agreement with experimental evidence, three of the associated gauge fields, W^\pm and Z^0 , need to have non-null mass, unlike the photon. This is achieved in the physical model by the Higgs-Kibble mechanism, in which the gauge symmetry is spontaneously broken to the electromagnetic gauge group $U(1)$. The resulting theory, which is sketched in Section 4.1, is unitary and renormalizable, and therefore suitable for perturbative calculations. Besides their intrinsic theoretical interest, massive Yang-Mills theories can also be useful in the construction of effective theories of strong interactions at low energies [49, 50].

Despite of the extraordinary success of the Standard Model, for the time being the Higgs boson, the footprint of the spontaneous symmetry breaking mechanism, has not been confirmed to exist. The Large Hadron Collider is expected to experimentally determine this.

Section 4.2 is dedicated to the Stueckelberg model. It is based on the introduction in the theory of scalar fields behaving exactly like gauge group parameters under gauge transformations, and whose kinematical term in the Lagrangian is a NLSM. The Minimal coupling prescription corresponding to these fields automatically provides mass terms for the gauge vector bosons, maintaining explicit gauge invariance and with no need for a Higgs particle. In spite of that, conventional Canonical Quantization treatment of these models seems to lead to a dichotomy between unitarity and renormalizability. We feel that this analysis might be incorrect, as it is founded in a wrong, implicit assumption about the symmetry of the NLSM.

Section 4.3 shows the general framework for GAQ to deal with field theories and Section 4.4 analyzes a Stueckelberg model based on a “partial-trace” sigma-model Lagrangian on an orbit of the gauge group G , G/H . The minimal coupling of the new (Goldstone-like) scalar bosons provides mass terms to those intermediate vector bosons associated with the quotient G/H , while the H -vector potentials remain massless. The main virtue of a partial trace on G/H , rather than on the entire G , is that we can find an

infinite-dimensional symmetry, with nontrivial Noether invariants, which ensures quantum integrability in our quantization scheme. Then, we analyze this symmetry group of massive Yang-Mills theories and their quantization. On the quantum representation space of this extended symmetry group, a quantum Hamiltonian preserving the representation can be given, whose classical analog reproduces the equations of motion.

We end up the chapter in Section 4.5, where the present formalism is applied to the case $G = SU(2) \times U(1)$, as a Higgsless alternative to the Standard Model of electroweak interactions.

It is not within the scope of this thesis work to deal with conventional perturbative computations which would allow a quantitative comparison with usual results in quantum field theory, but to establish foundations to which they could be fixed rigorously, pointing to weaknesses of previous analyses. In Chapter 5 we set up a possible approach to this kind of calculations: we explore the $O(N)$ -invariant Non-Linear Sigma Model (NLSM) in a different perturbative regime from the usual (relativistic) one, attempting to take advantage of the knowledge of non-canonical basic commutation relations adapted to the underlying $O(N)$ symmetry of the system, which also account for the non-trivial (non-flat) geometry and topology of the target manifold.

Changing the orientation of previous chapters, the purpose of Chapter 6 is to throw some light on the subject of quantum dissipation with the guide of symmetry. In Section 6.1, we generalize to the quantum regime the classical Arnold transformation, by which all dynamical systems whose classical equations of motion are non-homogeneous linear second-order ordinary differential equations, including systems with friction linear in velocity, can be related to free particle dynamical systems. By using the quantum Arnold transformation (QAT), we import basic operators from the free particle system, which satisfy the condition of being integrals of the motion and close a Heisenberg-Weyl algebra. Also, the complete set of symmetries of the quantum free particle, the Schrödinger group, can be realized on the Caldirola-Kanai model of the damped harmonic oscillator, providing as many conserved quantities as in the free particle. In addition, the transformation turns out to be extremely useful to compute objects that would otherwise need laborious calculations, such as wave functions, the quantum propagator or the evolution operator.

With the aid of the QAT between the free particle and the harmonic oscillator, an infinite discrete series of solutions of the free Schrödinger equation in one dimension is constructed in Section 6.2. These solutions are normalizable, expand the whole space of solutions, are spatially multi-localized and are eigenstates of a suitable defined number operator. Associated with these states new sets of coherent states for the free particle are defined representing traveling multi-localized wave packets. These states are also constructed in higher dimensions, leading to the quantum mechanical version of the Hermite-Gauss and Laguerre-Gauss states of paraxial wave optics.

Time translations in the non-free system do not belong to the imported Schrödinger group from the free particle. This is to be expected, as the classical equation of motion includes a friction term and the energy in this system is not conserved. The following question immediately arises: Is there any finite-dimensional group of symmetry containing time translations and, at least, the basic operators? The answer is 'yes', and Section 6.3

pays attention to this question in the case of the damped harmonic oscillator and the surprising consequences of the subsequent calculation: for this symmetry to act properly, it is necessary to enlarge the physical system with a new degree of freedom, corresponding to a new particle with interesting properties. This could be understood as a very simple version of the gauge principle, in which a bigger symmetry for the original “free” system is imposed. In fact, this new system with two degrees of freedom *is* the Bateman’s dual system. Taking advantage of the symmetry approach, we will go a bit further and provide a group law corresponding to the symmetries of the dual system.

With the light of this group law, in Section 6.4 we give an analysis of the quantization of the dual system that we have encountered. In particular, we show that it is possible to find a first-order Schrödinger equation, from which the wave functions and the energy spectrum can be obtained, as well as the more usual second-order equation. We also illustrate how the Caldirola-Kanai system can be recovered by means of a constraint process.

This dissertation is heavily based on the following published papers and in preparation, and a great part of the text is extracted from them.

- V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *Quantum Integrability of the Dynamics on a Group Manifold*, J. Nonlinear Math. Phys. **15**, 1 (2008).
- M. Calixto, V. Aldaya, F.F. López-Ruiz and E. Sánchez-Sastre, *Coupling Nonlinear Sigma-Matter to Yang-Mills Fields: Symmetry Breaking Patterns*, J. Nonlinear Math. Phys. **15**, 91 (2008).
- V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *Group-quantization of non-linear sigma models: particle on S^2 revisited*, Rep. Math. Phys. **64**, 49-58 (2009).
- J. Guerrero, F.F. López-Ruiz, M. Calixto and V. Aldaya, *On the geometry of the phase spaces of some $SO(2,1)$ invariant systems*, Rep. Math. Phys. **64**, 329-340, (2009).
- V. Aldaya, M. Calixto and F.F. López-Ruiz, *A Quantizable Model of Massive Gauge Vector Bosons without Higgs*, Mod. Phys. Lett. A **24**, No. 34, 2731-2740, (2009).
- V. Aldaya, M. Calixto and F.F. López-Ruiz, *Non-Canonical Perturbation Theory of Non-Linear Sigma Fields*, accepted for publication on Oct 18, 2010, in Mod. Phys. Lett. A, arXiv:1007.2820.
- V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz, *The quantum Arnold transformation*, accepted for publication in J. Phys. A; arXiv:1010.5521.
- V. Aldaya, M. Calixto and F. F. López-Ruiz, *Symmetry group for massive Yang-Mills theories*, in preparation.
- V. Aldaya, M. Calixto, J. Guerrero and F. F. López-Ruiz, *Symmetries of Non-Linear Systems: Group Approach to their Quantization*, in preparation.
- J. Guerrero, F.F. López-Ruiz, V. Aldaya and F. Cossío, *Discrete basis of localized quantum states for the free particle*; arXiv:1010.5525.

- V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz, *A symmetry trip from Caldirola to Bateman damped systems*, in preparation.

Fundamentals of Group Approach to Quantization

2.1 The postulate of quantization

The formalism of Quantum Mechanics is frequently introduced in textbooks through the enumeration of a series of postulates, collecting the amount of physical insight about the quantum world (see for instance the book by Galindo and Pascual [1]). We are instructed in the description of physical systems in terms of pure states represented by unit rays in a complex Hilbert space \mathcal{H} and observables represented by self-adjoint operators in \mathcal{H} ; in the probability of obtaining a given value when measuring an observable and the subsequent collapse of the wave function; in the time evolution of the states described by the Schrödinger equation; and finally, in a way to construct the self-adjoint operators corresponding to the physical observables by means of the canonical quantization rules. Clarify and formalize this last postulate of quantization rigorously is the main purpose of Group Approach to Quantization (GAQ). The slogan that might summarize GAQ is “represent a *true* symmetry group irreducibly and unitarily”, instead of “canonically quantize”. At the same time, GAQ achieves a conceptual generalization of the Schrödinger equation, which is viewed as a *polarization* equation in the scheme.

The postulate of quantization formulated in [1] reads:

- Postulate VI. For a physical system in which the Cartesian coordinates are q_1, q_2, \dots, q_N , with corresponding conjugate momenta p_1, p_2, \dots, p_N , the operators \hat{X}_r and \hat{P}_s , which represent these observables in Quantum Mechanics, must satisfy the commutation relations

$$[\hat{X}_r, \hat{X}_s] = 0, \quad [\hat{P}_r, \hat{P}_s] = 0, \quad [\hat{X}_r, \hat{P}_s] = i\hbar\delta_{rs}.$$

If the system has an observable whose classical expression is $A(q_1, \dots, q_N, p_1, \dots, p_N; t)$, in usual applications of Quantum Mechanics the corresponding operator is obtained from this expression, conveniently written, by substituting the operators \hat{X}_r and \hat{P}_s for the variables q_r and p_s , respectively.

Thus, Canonical Quantization intends to transfer the Poisson structure of the Classical Hamiltonian mechanics to the “quantum representation” of the physical system. As already mentioned in the Introduction, this presupposes a symmetry of the classical solution manifold of the system that might not be there in the case of non-linear systems.

This prejudgment is in part induced by the Darboux-Weinstein theorem of classical mechanics [3], which states that any two symplectic manifolds of the same dimension are locally symplectomorphic to each other, and therefore to a flat one.

Even when flatness of the solution manifold comes to help, so that the applicability of Canonical Quantization might seem undoubted, its topology must be taken into account. A good example is the application of the canonical rule to a particle moving in a half line: we are not able to describe all physical states. This system possesses a bound state that is usually discarded on behalf of the self-adjointness of canonically commuting basic position and momentum operators [13]. One must wonder whether a similar pathology might happen when blindly applying the postulate of Canonical Quantization.

It should also be noted the phrase “conveniently written” in the Postulate VI when referring to the obtention of operators representing arbitrary functions of the basic coordinates. It reflects the ambiguity encompassed in this method of quantization. Any function of the basic variables can be, in principle, quantizable, with the restriction of arriving at a self-adjoint operator. There is not a general way to single out quantizable functions nor a method to obtain their quantized version, and the specific expressions are justified ultimately by experimental evidence. It is sometimes believed that one needs only to generate the correspondence of these basic variables to the basic operators, or even the correspondence between their quadratic versions, and that the operator status of other (derived) observables will be automatically achieved. Combinations such as $q_i p_j$ are represented symmetrizing the two orderings but this supplemented rule however causes problem. If ones tries to built operator for higher order combinations, one finds that it is not possible to keep this rule consistently. Consider the Poisson bracket identity in one dimension

$$\frac{1}{9}\{q^3, p^3\} = q^2 p^2 = \frac{1}{3}\{q^2 p, p^2 q\}.$$

If we call $\hat{\mathcal{Q}}(f(q, p))$ the quantum operator corresponding to the function $f(q, p)$, the left hand side of this equation leads to

$$\frac{1}{9}[\hat{\mathcal{Q}}(q^3), \hat{\mathcal{Q}}(p^3)] = \hat{\mathcal{Q}}(q)^2 \hat{\mathcal{Q}}(p)^2 - 2i\hbar \hat{\mathcal{Q}}(q) \hat{\mathcal{Q}}(p) - \frac{2}{3}\hbar^2,$$

while its right hand side gives

$$\frac{1}{9}[\hat{\mathcal{Q}}(q^2 p), \hat{\mathcal{Q}}(p^2 q)] = \hat{\mathcal{Q}}(q)^2 \hat{\mathcal{Q}}(p)^2 - 2i\hbar \hat{\mathcal{Q}}(q) \hat{\mathcal{Q}}(p) - \frac{1}{3}\hbar^2.$$

This shows that one could not consistently quantize observables beyond quadratic combinations of q's and p's, and it is the basic result of the theorems by Groenwald and van Hove [2, 3, 4].

Geometric Quantization [8, 9, 10, 11] attempted to solve the situation, reformulating the ambiguous algorithm to provide some mathematical soundness. It succeeded in bringing the quantum theory to a geometrical language, proposing a coordinate-free framework. However, it finds severe problems in reducing the representation given by the pre-quantization scheme, or even unitarity problems in the reduced representation.

2.2 The role of central extensions of “classical” symmetries

Let us consider the symmetry of the Lagrangian of the free particle in 1+1 dimensions:

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2$$

Under the classical Galilean transformations

$$x' = x + A + Vt, \quad t' = t + B, \quad (2.1)$$

the Lagrangian moves to $\mathcal{L}' = \frac{1}{2} m (\dot{x} + V)^2 = \mathcal{L} + \frac{d}{dt} (\frac{1}{2} m V^2 t + m V x)$. That is, \mathcal{L} is not strictly invariant, but *semi-invariant*, due to the presence of the total derivative.

In infinitesimal terms something similar happens. Taking the Lie derivative of \mathcal{L} with respect to the generators of the group results in:

$$X_B = \frac{\partial}{\partial t} \quad \Rightarrow \quad X_B \mathcal{L} = 0$$

$$X_A = \frac{\partial}{\partial x} \quad \Rightarrow \quad X_A \mathcal{L} = 0$$

$$X_V = t \frac{\partial}{\partial x} + \frac{\partial}{\partial \dot{x}} \Rightarrow X_V \mathcal{L} = \frac{d}{dt} (m x) \neq 0$$

The same is also valid for the *Poincaré-Cartan form*:

$$\Theta_{PC} = p dx - H dt = \frac{\partial \mathcal{L}}{\partial \dot{x}} dx - (\dot{x} p - \mathcal{L}) dt = \frac{\partial \mathcal{L}}{\partial \dot{x}} (dx - \dot{x} dt) + \mathcal{L} dt$$

whose Lie derivative is:

$$L_{X_B} \Theta_{PC} = 0, \quad L_{X_A} \Theta_{PC} = 0, \quad L_{X_V} \Theta_{PC} = d(m x) \neq 0. \quad (2.2)$$

The quantum free particle suffers from the same “pathology” although it manifests in a different manner. Let us apply the Galilean transformations (2.1) to the Schrödinger equation. We get:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi \rightarrow i\hbar \frac{\partial \Psi}{\partial t'} = -\frac{\hbar^2}{2m} \nabla'^2 \Psi - i\hbar V \frac{\partial \Psi}{\partial x'}.$$

The extra term can be compensated if we also transform the wave function by means of a non-trivial phase:

$$\Psi \rightarrow \Psi' = e^{i \frac{m}{\hbar} (Vx + \frac{1}{2} V^2 t)} \Psi. \quad (2.3)$$

Then, we recover the original (fully primed) Schrödinger equation $i\hbar \frac{\partial \Psi'}{\partial t'} = -\frac{\hbar^2}{2m} \nabla'^2 \Psi'$. Joining together the Galilean transformations (2.1) and the phase transformation (2.3)

we obtain a group of *strict* symmetry whose group law is:

$$\begin{aligned}
B'' &= B' + B \\
A'' &= A' + A + V' B \\
V'' &= V' + V \\
\zeta'' &= \zeta' \zeta e^{i \frac{m}{\hbar} [A' V + B(V' V + \frac{1}{2} V'^2)]},
\end{aligned} \tag{2.4}$$

where the last line has the general form $\zeta'' = \zeta' \zeta e^{i \frac{m}{\hbar} \xi(g', g)}$, with $\zeta \equiv e^{i\phi} \in U(1)$ and the function ξ being that which is customarily named 2-cocycle on the Galilei group, characterized by the mass m [20, 23, 24]. A constant \hbar with the dimensions of an action has to be introduced to keep the exponent dimensionless.

The infinitesimal version of the group law (2.4) is expressed by means of the extended Lie algebra commutators:

$$[\tilde{X}_B, \tilde{X}_A] = 0, \quad [\tilde{X}_B, \tilde{X}_V] = \tilde{X}_A, \quad [\tilde{X}_A, \tilde{X}_V] = -m \tilde{X}_\phi. \tag{2.5}$$

One of the relevant points concerning both the *strict invariance* and, consequently, the *centrally extended symmetry* is that the corresponding *extended Lie algebra* now properly represents the Poisson algebra generated by $\langle H \equiv \frac{p^2}{2m}, K \equiv x - \frac{p}{m} t, P \equiv p, 1 \rangle$ when acting as ordinary derivations on complex functions, provided that we impose that the new generator \tilde{X}_ϕ acts on Ψ as $\tilde{X}_\phi \Psi = i\Psi$, or, in finite terms, $\Psi(\zeta g) = \zeta \Psi(g)$. Notice that the unextended algebra, with the commutator $[X_A, X_V] = 0$, is not an isomorphic image of the corresponding Noether invariants H, P, K algebra.

There is yet another remarkable advantage of requiring the strict symmetry of a given arbitrary classical system. In fact, such a symmetry can only be realized faithfully if we extend the classical phase space M parameterized by K, P (or solution manifold) by an extra variable, to be identified with ϕ or $\zeta = e^{i\phi}$. In the compact ($U(1)$) case, that is, the choice of ζ , we thus arrive at the notion of a *quantum manifold* Q [8, 11]. In this manifold, locally parameterized by $K, P, \zeta \equiv e^{i\phi}$, an extended Liouville form (Θ_{PC} defines the Liouville form ϑ on the solution manifold except for a total differential)

$$\Theta = \vartheta + \frac{d\zeta}{i\zeta} \quad (\text{or } PdK + d\phi_0)$$

substitutes successfully the ordinary one in the search for an invertible duality between Hamiltonian functions and Hamiltonian vector fields. In fact, the Hamiltonian correspondence

$$f \mapsto X_f \text{ such that } i_{X_f} d\vartheta = -df$$

has the real numbers \mathbb{R} as kernel. However the correspondence

$$f \mapsto \tilde{X}_f \text{ such that } i_{\tilde{X}_f} d\Theta = -df, i_{\tilde{X}_f} \Theta = f$$

has unique solution.

This is, so to speak, the starting point for GQ, where the pair (symplectic manifold) $(M, \omega \equiv d\vartheta)$ is replaced with the pair (Q, Θ) as a $U(1)$ -principal bundle with connection (quantum manifold) under the requirement that the curvature of Θ defines on M the symplectic form ω with integer co-homology class (tantamount to say that the integration of ϑ on closed curves results in an integer; this is a modern, geometric version of the Bohr-Sommerfeld rules [11]). The association $f \mapsto \hat{f} \approx \tilde{X}_f$ defines the pre-quantum operators, as derivations on complex $U(1)$ -functions on Q , which realize a unitary representation of the Poisson bracket although non-irreducible. The true quantization, that is to say, the irreducibility, is intended to be achieved after the polarization condition is imposed (see Refs. [8, 11] and the analogous condition in next section).

It should be mentioned that the possibility exists of extending the classical phase space by the real line and the classical group by the non-compact additive group \mathbb{R} . In that case the constant \hbar is no longer needed and the resulting theory describes the classical limit in a *global* version of the Hamilton-Jacobi formulation (see Ref. [7]).

2.3 Group Approach to Quantization

The essential idea underlying a group-theoretical framework for quantization consists in selecting a given subalgebra $\tilde{\mathcal{G}}$ of the classical Poisson algebra including $\langle H, p_i, x^j, 1 \rangle$ and finding its *unitary irreducible representations* (unirreps), which constitute the possible *quantizations*. Although the actual procedure for finding unirreps might not be what really matters from the physical point of view we proceed along a well-defined algorithm, the group approach to quantization or GAQ for brief, to obtain them for any Lie group.

All the ingredients of GAQ are canonical structures defined on Lie groups and the very basic ones consist in the *two* mutually *commuting* copies of the Lie algebra $\tilde{\mathcal{G}}$ of a group \tilde{G} of *strict symmetry* (of a given physical system), that is, the set of left- and right-invariant vector fields:

$$\mathcal{X}^L(\tilde{G}) \approx \tilde{\mathcal{G}} \approx \mathcal{X}^R(\tilde{G})$$

in such a way that one copy, let us say $\mathcal{X}^R(\tilde{G})$, plays the role of *pre-Quantum Operators* acting (by usual derivation) on complex (wave) functions on \tilde{G} , whereas the other, $\mathcal{X}^L(\tilde{G})$, is used to *reduce* the representation in a manner *compatible* with the action of the operators, thus providing the *true quantization*.

In fact, from the group law $g'' = g' * g$ of any group \tilde{G} , we can read two different left- and right-actions:

$$g'' = g' * g \equiv L_{g'} g, \quad g'' = g' * g \equiv R_g g'. \quad (2.6)$$

Both actions commute and so do their respective generators \tilde{X}_a^R and \tilde{X}_b^L , i.e. $[\tilde{X}_a^L, \tilde{X}_b^R] = 0 \forall a, b$.

Another manifestation of the commutation between left and right translations corresponds to the invariance of the left-invariant canonical 1-forms, $\{\theta^{L^a}\}$ (dual to $\{\tilde{X}_b^L\}$, i.e. $\theta^{L^a}(\tilde{X}_b^L) = \delta_b^a$) with respect to the right-invariant vector fields, that is: $L_{\tilde{X}_a^R} \theta^{L^b} = 0$ and the

other way around ($L \leftrightarrow R$). In particular, we dispose of a natural *invariant volume* ω (the Haar measure) on the group manifold since we have:

$$L_{\tilde{X}_a^R}(\theta^{L^b} \wedge \theta^{L^c} \wedge \theta^{L^d} \dots) \equiv L_{\tilde{X}_a^R} \omega = 0. \quad (2.7)$$

We should then be able to recover all *physical ingredients* of quantum systems out of *algebraic structures*. In particular, the Poincaré-Cartan form Θ_{PC} and the phase space itself $M \equiv (x^i, p_j)$ should be regained from a group of *strict symmetry* \tilde{G} . In fact, in the special case of a Lie group which bears a central extension with structure group $U(1)$ parameterized by $\zeta \in C$ such that $|\zeta|^2 = 1$, as we are in fact considering, the group manifold \tilde{G} itself can be endowed with the structure of a *principal bundle* with an *invariant connection*, thus generalizing the notion of *quantum manifold*.

More precisely, the $U(1)$ -component of the left-invariant canonical form (dual to the *vertical* generator \tilde{X}_ζ^L , i.e. $\theta^{L(\zeta)}(\tilde{X}_\zeta^L) = 1$) will be named *quantization form* $\Theta \equiv \theta^{L(\zeta)}$ and generalizes the Poincaré-Cartan form Θ_{PC} of Classical Mechanics. The quantization form remains *strictly invariant* under the group \tilde{G} in the sense that

$$L_{\tilde{X}_a^R} \Theta = 0 \quad \forall a$$

whereas Θ_{PC} is, in general, only *semi-invariant*, that is to say, it is invariant except for a total differential.

It should be stressed that the construction of a true quantum manifold in the sense of Geometric Quantization [8, 9] can be achieved by taking in the pair $\{\tilde{G}, \Theta\}$ the quotient by the action of the subgroup generated by those left-invariant vector fields in the kernel of Θ and $d\Theta$, that which is called in mathematical terms *characteristic module* of the 1-form Θ ,

$$\mathcal{C}_\Theta \equiv \{\tilde{X}^L / i_{\tilde{X}^L} d\Theta = 0 = i_{\tilde{X}^L} \Theta\}.$$

A further quotient by the structure subgroup $U(1)$ provides the *classical solution Manifold* M or classical *phase space*. Even more, the vector fields in \mathcal{C}_Θ constitute the (generalized) *classical equations of motion*.

On the other hand, the right-invariant vector fields are used to provide classical functions on the phase space. In fact, the functions

$$F_a \equiv i_{\tilde{X}_a^R} \Theta \quad (2.8)$$

are stable under the action of the left-invariant vector fields in the characteristic module of Θ , the equations of motion,

$$L_{\tilde{X}^L} F_a = 0 \quad \forall \tilde{X}^L \in \mathcal{C}_\Theta$$

and then constitute the *Noether invariants*.

As a consequence of the central extension structure in \tilde{G} the Noether invariants (and the corresponding group parameters) are classified in basic and non-basic (or evolutive) depending on whether or not the corresponding generators produce the central generator by commutation with some other. Basic parameters (Noether invariants) are paired (and

independent). Non-basic Noether invariants (like energy or angular momenta) can be written in terms of the basic ones (positions and momenta).

As far as the quantum theory is concerned, the above-mentioned quotient by the classical equations of motion is really not needed. We consider the space of complex functions Ψ on the whole group \tilde{G} and restrict them to only $U(1)$ -functions, that is, those which are homogeneous of degree 1 on the argument $\zeta \equiv e^{i\phi} \in U(1)$, or in infinitesimal terms

$$\tilde{X}_\phi^L \Psi = i\Psi. \quad (2.9)$$

On these functions the right-invariant vector fields act as *pre-quantum operators* by ordinary derivation. They are, in fact, Hermitian operators with respect to the scalar product with measure given by the invariant volume ω defined above (2.7). However, this action is not a proper quantization of the Poisson algebra of the Noether invariants (associated with the symplectic structure given by $d\Theta$) since there is a set of non-trivial operators commuting with this representation. In fact, all the left-invariant vector fields do commute with the right-invariant ones, i.e. the pre-quantum operators. According to Schur's Lemma those operators must be trivialized to reduce the representation. To this end we define a polarization subalgebra as follows:

A polarization \mathcal{P} is a maximal left subalgebra containing the characteristic subalgebra \mathcal{G}_Θ and excluding the central generator.

The role of a polarization is that of *reducing* the representation which then constitutes a true *quantization*. To this end we impose on wave functions the polarization condition:

$$\tilde{X}_b^L \Psi = 0 \quad \forall \tilde{X}_b^L \in \mathcal{P}.$$

In finite terms the polarization condition is expressed by the invariance of the wave functions under the finite action of the Polarization Subgroup G_P acting from the right, that is:

$$\Psi(g' g_P) = \Psi(g') \quad \forall g_P \in G_P. \quad (2.10)$$

To be intuitive, a polarization is made of half the left-invariant vector fields associated with basic (independent) variables of the solution manifold in addition to those associated with evolutive parameters as time or rotational angles. We should remark that the classification above-mentioned of the Noether invariants in basic and non-basic also applies to the quantum operators so that the latter ones are written in terms of the formers.

As an additional comment regarding polarization conditions, it must be stressed that when expressed as quantum equations, they contain, in particular, the evolution equation properly, that is, the Schrodinger(-like) equation. In this respect these polarizations (and the GAQ method itself) depart from those in Geometric Quantization, which are imposed only after having taken the quotient by the classical evolution explicitly, that which means having solved the classical equations. Another respect on which GAQ departs from GQ is in that the entire enveloping algebra (both left and right ones) can be used to construct higher-order Polarizations and higher-order operators.

The integration volume ω can be restricted to the Hilbert space of polarized wave functions \mathcal{H} by means of a canonical procedure a bit technical for the scope of this work. We refer the reader to Ref. [25].

Before ending this section let us mention that the existence of a polarization containing the entire characteristic subalgebra (usually referred to as full polarization) is not guaranteed in general and we then can resort to the left enveloping algebra to complete the polarization in the same way that any operator in the right enveloping algebra can be properly realized as a quantum operator (see Ref. [26]). Higher-order polarizations are used by strict necessity, when no full polarization can be found (in this case the system is *anomalous* in the standard physical sense [27]), or simply by pure convenience of realizing the quantization in a particular "representation" adapted to given variables.

2.4 The Free Galilean Particle

We shall adopt the notation $B \equiv t$, $A \equiv x$, $V \equiv v$ ($p \equiv m v$) for the parameters in the group law to reinforce the fact that all physical variables do emerge naturally from the group manifold itself and the dimension will be kept to $1 + 1$ to reduce the expressions to the minimum.

Reading the group law (2.4) in the new variables and deriving the double primed variables with respect to every non-primed and primed one at the identity we get the explicit expressions of the left- and right-vector fields, respectively:

$$\begin{aligned} \tilde{X}_t^L &= \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{1}{2} m v^2 \frac{\partial}{\partial \phi} & \tilde{X}_t^R &= \frac{\partial}{\partial t} \\ \tilde{X}_x^L &= \frac{\partial}{\partial x} & \tilde{X}_x^R &= \frac{\partial}{\partial x} + m v \frac{\partial}{\partial \phi} \\ \tilde{X}_v^L &= \frac{\partial}{\partial v} + m x \frac{\partial}{\partial \phi} & \tilde{X}_v^R &= \frac{\partial}{\partial v} + t \frac{\partial}{\partial x} + m t v \frac{\partial}{\partial \phi} \\ \tilde{X}_\phi^L &= \frac{\partial}{\partial \phi} & \tilde{X}_\phi^R &= \frac{\partial}{\partial \phi} \end{aligned} \quad (2.11)$$

By duality on the left generators, selecting the $U(1)$ component, or by using the direct formula

$$\Theta \equiv \theta^{L\phi} = \sum_i \frac{\partial \phi''}{\partial g^i} \Big|_{g'=g^{-1}} dg^i, \quad (2.12)$$

one can compute the quantization form (the actual expression of the Poincaré-Cartan part is defined up to a total differential depending of the particular co-cycle used in the group law, which is defined in turns up to a co-boundary; see Ref.[20, 7]):

$$\Theta \equiv \theta^{L\phi} = -m x d v - \frac{1}{2} m v^2 d t + d \phi$$

From the commutation relations (the left ones change the structure constants by a global sign)

$$[\tilde{X}_t^R, \tilde{X}_x^R] = 0, \quad [\tilde{X}_t^R, \tilde{X}_v^R] = \tilde{X}_x^R, \quad [\tilde{X}_x^R, \tilde{X}_v^R] = -m \tilde{X}_\phi^R \quad (2.13)$$

one rapidly identifies x , v as canonically conjugated (dynamical) variables and t as a kinematical parameter. In fact, the left generator \tilde{X}_t^L generates the characteristic subalgebra \mathcal{G}_Θ and constitutes the classical equations of motion (generalized, since there is an extra equation for the central parameter).

The quantum wave functions are complex functions on \tilde{G} , $\Psi = \Psi(\zeta, x, v, t)$, restricted by the $U(1)$ -function condition (2.9), as well as the polarization conditions

$$\tilde{X}_a^L \Psi = 0 \quad (a = t, x \text{ maximal set})$$

We then obtain:

$$\tilde{X}_\phi^L \Psi = i\Psi \Rightarrow \Psi = \zeta \Phi(t, x, v)$$

$$\tilde{X}_x^L \Psi = 0 \Rightarrow \Phi \neq \Phi(x), \Phi = \varphi(t, v)$$

$$\tilde{X}_t^L \Psi = 0 \Rightarrow \frac{\partial \varphi}{\partial t} + \frac{i}{2} m v^2 \varphi = 0 \Rightarrow i \frac{\partial \varphi}{\partial t} = \frac{p^2}{2m} \varphi,$$

i.e. the Schrödinger equation in momentum space.

On the (reduced) wave functions the right-invariant vector fields act reproducing the standard quantum operators in momentum space "representation":

$$\tilde{X}_x^R \varphi = m v \varphi, \quad \tilde{X}_v^R \varphi = \frac{\partial}{\partial v} \varphi, \quad \tilde{X}_t^R \varphi = -i \frac{p^2}{2m} \varphi, \quad (2.14)$$

the operator $\hat{E} \equiv i \tilde{X}_t^R$ being a function of the basic one $\hat{p} \equiv -i m \tilde{X}_x^R$. Had we considered the motion in 3 + 1 dimensions, we would have found also new operators in the characteristic subalgebra associated with rotations acquiring the usual expressions in terms of the basic operators \hat{v} and $\hat{x} \equiv -i \tilde{X}_v^R$.

Chapter 3

Mechanical models

3.1 Revisited Minimal Coupling Principle

In this section we attempt to describe group-theoretically the motion of a particle subjected to an external field. Even though we do not intend to account for the field degrees of freedom, the transformation properties of its “zero-modes” can be encoded into part of a symmetry group. The general mechanism under which a free particle starts suffering an interaction parallels the well-known Minimal Coupling Principle, which is now revisited from our group-theoretical approach. We shall be concerned here with the classical domain only.

Let \tilde{G} be a quantization group generated by $\{\tilde{X}_A\}$, $A = 1, \dots, n$ and $\{\tilde{X}_a\}$, $a = 1, \dots, m < n$ an invariant subalgebra:

$$[\tilde{X}_A, \tilde{X}_a] = C_{Aa}^b \tilde{X}_b$$

If we make “local” the subgroup generated by $\{\tilde{X}_a\}$, that is to say, if the corresponding group variables are allowed to depend arbitrarily on the space-time parameters, we get an infinite-dimensional Lie algebra:

$$\{f^a \otimes \tilde{X}_a, \tilde{X}_A\}$$

with the following new commutators:

$$\begin{aligned} [\tilde{X}_A, f^a \otimes \tilde{X}_a] &= f^a \otimes [\tilde{X}_A, \tilde{X}_a] + L_{\tilde{X}_A} f^a \otimes \tilde{X}_a \\ &= f^a \otimes C_{Aa}^b \tilde{X}_b + L_{\tilde{X}_A} f^a \otimes \tilde{X}_a \end{aligned} \quad (3.1)$$

Now we just attempt to “quantize” this new (local) group $\tilde{G}(\vec{x}, t)$.

3.1.1 Particle in an Electromagnetic Field.

We start from the $U(1)$ -extended Galilei group, \tilde{G} , and make the rigid group $\zeta = e^{i\phi} \in U(1)$ into “local”, i.e. we allow the parameter to depend on the space-time variables, $\phi = \phi(\vec{x}, t)$. The idea is to keep the invariance of the generalized Poincaré-Cartan form $\Theta = p_i dx^i - \frac{\vec{p}^2}{2m} dt + d\phi$ under the locally extended Galilei group.

According to the *Revisited Minimal Coupling Principle* [28] we only have to compute the 1-form Θ associated with the Galilei group extended by the infinite dimensional group $U(1)(\vec{x}, t)$. But, in order to parameterize properly the quantization group let us formally write

$$\phi(\vec{x}, t) = \phi(0, 0) + \phi_\mu(\vec{x}, t)x^\mu \equiv \phi + A_\mu(\vec{x}, t)x^\mu$$

and compute the group law:

$$t'' = t' + t$$

$$\vec{x}'' = \vec{x}' + R'\vec{x} + \vec{v}'t$$

$$\vec{v}'' = \vec{v}' + R'\vec{v}$$

$$A''_{\vec{x}} = A'_{\vec{x}} + R'A_{\vec{x}}$$

$$A''_t = A'_t + A_t + \vec{v}' \cdot R'A_{\vec{x}}$$

$$\phi'' = \phi' + \phi + \mathbf{m}[\vec{x}' \cdot R'\vec{v} + t(\vec{v}' \cdot R'\vec{v} + \frac{1}{2}v'^2)] + \mathbf{q}[\vec{x}' \cdot R'A_{\vec{x}} + t\vec{v}' \cdot R'A_{\vec{x}} + tA'_t]$$

where two different co-cycles characterized by m and q , that is, the mass and the electric charge, have been introduced.

From now on we shall disregard the rotation subgroup although the vector character of the variables will be maintained. Also, and since we do not intend to describe quantum aspects, the expression of the left-invariant generators will be omitted (see Ref. [28]) and only the Lie algebra commutators are written:

$$[\tilde{X}_t^L, \tilde{X}_{\vec{x}}^L] = 0 \quad [\tilde{X}_{x^i}^L, \tilde{X}_{A_{x^j}}^L] = \mathbf{q}\delta_{ij}\tilde{X}_\phi^L \quad [\tilde{X}_t^L, \tilde{X}_{\vec{v}}^L] = -\tilde{X}_{\vec{x}}^L \quad (3.2)$$

$$[\tilde{X}_{\vec{x}}^L, \tilde{X}_{A_t}^L] = 0 \quad [\tilde{X}_t^L, \tilde{X}_{A_x}^L] = 0 \quad [\tilde{X}_{\vec{v}}^L, \tilde{X}_{A_x}^L] = \tilde{X}_{A_t}^L \quad (3.3)$$

$$[\tilde{X}_t^L, \tilde{X}_{A_t}^L] = -\mathbf{q}\tilde{X}_\phi^L \quad [\tilde{X}_{\vec{v}}^L, \tilde{X}_{A_t}^L] = 0 \quad [\tilde{X}_{x^i}^L, \tilde{X}_{v^j}^L] = \mathbf{m}\delta_{ij}\tilde{X}_\phi^L \quad (3.4)$$

By duality from the explicit expression of the left-invariant generators, derived in turn from the group law, or directly from the composition law corresponding to the $U(1)$ parameter, through the formula (2.12), we obtain the quantization form

$$\Theta = -m\vec{x} \cdot d\vec{v} - q\vec{x} \cdot d\vec{A} - (\frac{1}{2}m\vec{v}^2 + qA_t)dt + d\phi$$

in the kernel of which we find generator of time evolution: X such that $i_X d\Theta = 0$, that is,

$$X = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \frac{q}{m} \left[\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) v^j + \frac{\partial A_0}{\partial x^i} + \frac{\partial A_i}{\partial t} \right] \frac{\partial}{\partial v_i}$$

which implies the following explicit equations of motion:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad m \frac{d\vec{v}}{dt} = q[\vec{v} \wedge (\vec{\nabla} \wedge \vec{A}) - \vec{\nabla} A_0 - \frac{\partial \vec{A}}{\partial t}]. \quad (3.5)$$

Making the standard change of variables

$$\vec{B} \equiv \vec{\nabla} \wedge \vec{A}, \quad \vec{E} \equiv -\vec{\nabla} A_0 - \frac{\partial \vec{A}}{\partial t}$$

we finally arrive at the ordinary equation of a particle suffering the Lorentz force:

$$m \frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \wedge \vec{B}]$$

As a last general comment, let us remark once again the physical relevance of central extensions. It might seem paradoxical the fact that a non-trivial vector potential (in the sense that it is not the gradient of a function) can be derived some how from the function $\phi(\vec{x}, t)$, but it is the central extension mechanism what insures that A_μ can be something different from the gradient of a scalar function. In other words, the generator X_{A_μ} in (3.4), for $q = 0$, necessarily generates trivial (gauge) changes in A_μ .

3.1.2 Particle in a gravitational field.

Let us pass very briefly through this finite-dimensional example where the computations are made in dimension $1 + 1$ although the vectorial notation is restored at the end. We start from the $U(1)$ -extended Poincaré group and make "local" the translation subgroup, the Lie algebra of which can be written as

$$[\tilde{X}_t^R, \tilde{X}_x^R] = 0 \quad [P_0, P] = 0 \quad (3.6)$$

$$[\tilde{X}_t^R, \tilde{X}_v^R] = \tilde{X}_x^R \quad \text{or} \quad [P_0, K] = P \quad (3.7)$$

$$[\tilde{X}_x^R, \tilde{X}_v^R] = -\frac{1}{c^2} \tilde{X}_t^R - m \tilde{X}_\phi^R \quad [P_0, K] = -\frac{1}{c^2} P_0 - m X_\phi, \quad (3.8)$$

and repeat the process of "localizing" translations in a way analogous to that followed for the $U(1)$ subgroup in the electromagnetic case. We formally write

$$f^\mu \otimes P_\mu = (f^\mu(0) + f^{\mu\sigma}(x)x_\sigma) \otimes P_\mu$$

and rename the functions $f^{\mu\nu}$ as $h^{\mu\sigma}$, which will prove to be the non-Minkowskian part of a non-trivial metric, that is: $h^{\mu\nu} \equiv g^{\mu\nu} - \eta^{\mu\nu}$

The Lie algebra must be explicitly written according to the general formula (3.1) and the rigid algebra (extended Poincaré). We show in boldface the terms that survive af-

ter an Inönü-Wigner contraction with respect to the subgroup generated by \tilde{X}_t (the non-relativistic limit):

$$\left[\tilde{X}_\nu, \tilde{X}_x \right] = -\tilde{X}_t + \mathbf{mc}\tilde{\mathbf{X}}_\phi \quad \left[\tilde{X}_t, \tilde{X}_{h_{0x}} \right] = \tilde{\mathbf{X}}_x \quad \left[\tilde{X}_\nu, \tilde{X}_t \right] = -\tilde{\mathbf{X}}_x \quad (3.9)$$

$$\left[\tilde{X}_x, \tilde{X}_{h_{0x}} \right] = -\tilde{X}_t - \mathbf{g}\tilde{\mathbf{X}}_\phi \quad \left[X_\nu, \tilde{X}_{h_{00}} \right] = -\tilde{X}_{h_{0x}} \quad \left[\tilde{X}_x, \tilde{X}_{h_{xx}} \right] = -\tilde{X}_x \quad (3.10)$$

$$\left[\tilde{X}_{h_{00}}, \tilde{X}_{h_{0x}} \right] = \tilde{X}_\nu \quad \left[X_\nu, \tilde{X}_{h_{xx}} \right] = \tilde{X}_{h_{0x}} \quad \left[\tilde{X}_{h_{0x}}, \tilde{X}_{h_{xx}} \right] = \tilde{X}_\nu \quad (3.11)$$

$$\left[\tilde{X}_t, \tilde{X}_{h_{00}} \right] = \tilde{X}_t + \mathbf{g}\tilde{\mathbf{X}}_\phi \quad \left[X_\nu, \tilde{X}_{h_{0x}} \right] = -\tilde{X}_{h_{00}} + \tilde{X}_{h_{xx}} \quad (3.12)$$

It should be remarked that we have naively written a *gravitational coupling constant* g in places that parallel those of the electric charge q in the Lie algebra that accounts for the electromagnetic interaction; that is to say, on the right hand side of the commutators $[\tilde{X}_t, \tilde{X}_{h_{00}}]$ and $[\tilde{X}_x, \tilde{X}_{h_{0x}}]$, but the Jacobi identity requires the equality $g = mc$, which may be properly identified with an algebraic version of the **Equivalence Principle**. Note that both q and $g = mc$ are true central charges (in the sense that they parameterize non-trivial central extensions) in the non-relativistic limit. To be precise, q also parameterizes non-trivial central extension in the Poincaré group, while g parametrizes just pseudo-extensions (see [73]).

Now, the group law must be computed order by order, although it is enough to keep the expansion up to the 3^{th} order for illustrating the dynamics. We remit the readers to Ref. [28] for a detailed computation and here only the final equation of motion are showed.

Geodesic Force: We introduce for simplicity the vector notation: $h^{0i} \equiv \vec{h}$. In terms of these variables the equations of motion, for low gravity and low velocity, are:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad m \frac{d\vec{v}}{dt} = -m \left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{h}) - \vec{\nabla} h^{00} - \frac{\partial \vec{h}}{\partial t} \right] + \frac{m}{4} \vec{\nabla} (\vec{h} \cdot \vec{h}).$$

They reproduce the standard geodesic motion, up to the limits mentioned, and in a form that emulate the electromagnetic motion (3.5) for the electromagnetic-like vector potential $\mathcal{A} = (h^{00} - \frac{1}{4} \vec{h} \cdot \vec{h}, \vec{h})$ according to that which is named “gravitoelectromagnetic” description in the literature (see, for instance, Ref. [29, 30, 31, 32]).

3.2 Simple “Non-Linear” Systems

As a preliminary step towards non-linearity, we shall consider the simpler example of the *free relativistic particle* with classical Lagrangian $\mathcal{L} = -mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}}$. As was previously commented, even though the corresponding equations of motion are linear, the “non-linearity” (non-quadratic, indeed) of the Lagrangian causes the impossibility of closing a finite-dimensional algebra containing H, x^i, p_j . Let us present the problem very quickly. The canonical momenta are derived in the standard way from \mathcal{L} :

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{m \dot{x}_i}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

as well as the Hamiltonian:

$$H = \dot{x}^i p_i - \mathcal{L} = \frac{m c^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} = \sqrt{\vec{p}^2 c^2 + m^2 c^4} \equiv p_0 c$$

where we make use of the traditional definition of four-momentum $\{p_\mu\} = \{p_0, p_i\}$.

Let us try to close the basic Poisson subalgebra including H . We find

$$\{x^i, p_j\} = \delta_j^i 1 \text{ (canonical Poisson bracket)}$$

$$\{H, p_i\} = 0$$

$$\text{but } \{H, x^i\} = -c \frac{p^i}{p_0} \left(\approx -\frac{p_i}{m} - \frac{1}{c^2} \frac{\vec{p}^2 p^i}{2m^3} + \dots \right)$$

So, the classical functions $\langle H, x^i, p_j, 1 \rangle$ *do not close* a finite Lie algebra. They “close” an algebra with structure constants *depending* on the energy $H \equiv p_0 c$.

Here, two different options arise. One option consists in trying to close an infinite-dimensional Poisson subalgebra by defining new functions which are quadratic and beyond in the basic functions x^i and p_j . This can be done order by order in the parameter $\frac{1}{c^2}$ which now plays the role of “coupling constant”. But there is another, far simpler when possible, which consists in looking for new “basic functions”, closing a finite-dimensional subalgebra, in terms of which we can rewrite the old basic functions. Here we proceed along this last line.

To this end we define the classical functions $k^i \equiv \frac{p_0}{m c} x^i$. Then, the new algebra $\langle H, p_i, k^j, J^k, 1 \rangle$ *does close* on the Poincaré algebra:

$$\{p_0, k^i\} = \{p_0, \frac{p_0}{m c} x^i\} = -\frac{p^i}{m}$$

$$\{p_0, p_i\} = 0$$

$$\{k^i, k^j\} = -\eta_{ij}^k \cdot \eta_{mn}^k x^m p^n \equiv -\eta_{ij}^k J^k$$

$$\{k^i, p_j\} = \left\{ \frac{p_0}{m c} x^i, p_j \right\} = \frac{p_0}{m c} \delta_j^i$$

If we quantize the Poincaré group, we regain the original (*infinite-dimensional*) algebra of quantum operators inside its enveloping algebra:

$$\langle \hat{H}, \hat{k}^i, \hat{x}^j \sim \frac{m c}{2} (\hat{p}_0^{-1} \hat{k}^j + \hat{k}^j \hat{p}_0^{-1}), \hat{J}^k, \hat{1} \rangle$$

To do that we parameterize a central extension of the Poincaré group by (abstract) variables $\{a^0, \vec{a}, \vec{v}, \vec{\epsilon}, \zeta\}$ so that the, let us say, right-invariant generators reproduce the respective functions $\{H, \vec{p}, \vec{k}, \vec{J}, 1\}$ as Noether invariants as well as the Poisson brackets above.

We are not going to present here the precise details of the quantization of the Poincaré group nor insist in those typical problems related to the “position operator” in quantum relativity that can be read from, for instance, Ref. [81]. Let us just comment that the *wave functions* are solutions of a higher-order Polarization $\mathcal{P}\psi = 0$, with

$$\mathcal{P} = \langle \tilde{X}_{a^0}^{LHO} \equiv (\tilde{X}_{a^0}^L)^2 - c^2(\tilde{X}_{\vec{a}}^L)^2 - \frac{2imc^2}{\hbar}\tilde{X}_{a^0}^L, \quad \tilde{X}_{\vec{v}}^L, \quad \tilde{X}_{\vec{e}}^L \rangle$$

from which we arrive at a wave function which depends only on a^0 and \vec{a} and satisfies the ordinary Klein-Gordon equation in variables x^0 and \vec{x} :

$$(\square + m^2)\psi = 0,$$

after the change $a^\mu = \frac{p_0}{mc}x^\mu$.

3.3 Particle moving on a group manifold: case of the $SU(2)$ group

Let us now adopt a similar point of view to that of the previous section, that is, we shall start from the classical Lagrangian and try to close a Poisson subalgebra containing $\langle H, q, p \rangle$. However, except for simple examples such a subalgebra is infinite. To keep ourselves in finite dimensions (that is, with a finite number of degrees of freedom) we may alternatively resort to an auxiliary, different finite-dimensional Poisson subalgebra (closing a group G) such that, in its enveloping algebra, the original functions $\langle H, q, p \rangle$ can be found, and therefore quantized. In fact, in the GAQ scheme, not only the generators of the original group G can be quantized, but also the entire universal enveloping algebra. This procedure has been explicitly achieved in dealing with the quantum dynamics of a particle in a (modified) Pöschl-Teller potential[6], where the “first-order” (auxiliary) group G used was $SL(2, \mathbb{R})$.

We start by parameterizing rotations with a vector $\vec{\varepsilon}$ in the rotation-axis direction and with modulus

$$|\vec{\varepsilon}| = 2\sin\frac{\varphi}{2}$$

$$R(\vec{\varepsilon})_j^i = \left(1 - \frac{\vec{\varepsilon}^2}{2}\right)\delta_j^i - \sqrt{1 - \frac{\vec{\varepsilon}^2}{4}}\eta_{jk}^i\varepsilon^k + \frac{1}{2}\varepsilon^i\varepsilon_j$$

In these coordinates the canonical left-invariant 1-forms read:

$$\theta_j^{L(i)} = \left[\sqrt{1 - \frac{\vec{\varepsilon}^2}{4}}\delta_j^i + \frac{\varepsilon^i\varepsilon_j}{4\sqrt{1 - \frac{\vec{\varepsilon}^2}{4}}} + \frac{1}{2}\eta_{jm}^i\varepsilon^m \right]$$

and in terms of these the *particle- σ -Model Lagrangian* acquires the following expression:

$$\mathcal{L} = \frac{1}{2}\delta_{ij}\theta_m^{L(i)}\theta_n^{L(j)}\dot{\varepsilon}^m\dot{\varepsilon}^n = \frac{1}{2}\left[\delta_{ij} + \frac{\varepsilon_i\varepsilon_j}{4(1 - \frac{\vec{\varepsilon}^2}{4})}\right]\dot{\varepsilon}^i\dot{\varepsilon}^j \equiv \frac{1}{2}g_{ij}\dot{\varepsilon}^i\dot{\varepsilon}^j$$

Proceeding much in the same way followed in the previous section, we compute the canonical momenta:

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}^i} = g_{ij} \dot{\varepsilon}^j$$

and the Hamiltonian:

$$\mathcal{H} = \pi_i \dot{\varepsilon}^i - \mathcal{L} = \frac{1}{2} g^{-1ij} \pi_i \pi_j$$

We assume the canonical bracket between the basic functions ε^i and π_j :

$$\{\varepsilon^i, \pi_j\} = \delta_j^i \quad \text{added with}$$

$$\{\mathcal{H}, \varepsilon^i\} = -g^{-1ij} \pi_j$$

$$\{\mathcal{H}, \pi_i\} = \frac{1}{2} (\vec{\varepsilon} \cdot \vec{\pi}) \pi_i,$$

so that $\langle \mathcal{H}, \varepsilon^i, \pi_j, 1 \rangle$ do not close a finite-dimensional Lie algebra.

However, we may define the following set of new "coordinates", "momenta" and even, "energy" and "angular momenta":

$$\langle p^i \equiv 2g^{-1ij} \pi_j, k^j \equiv \sqrt{2\mathcal{H}} \varepsilon^j, E \equiv 2\sqrt{2\mathcal{H}}, J^k \equiv \eta_{mn}^k \varepsilon^m \pi^n \rangle.$$

They close the Lie algebra of $SO(3,2)$ i.e. an Anti-de Sitter algebra. That is, the basic brackets:

$$\{E, p^i\} = k^i$$

$$\{E, k^j\} = -p^j$$

$$\{k^i, p_j\} = \delta_j^i E,$$

along with the induced ones:

$$\{k^i, k^j\} = -\eta_k^{ij} J^k$$

$$\{p^i, p^j\} = -\eta_k^{ij} J^k$$

$$\{J^i, J^j\} = \eta_k^{ij} J^k$$

$$\{J^i, k^j\} = \eta_k^{ij} k^k$$

$$\{J^i, p^j\} = \eta_k^{ij} p^k$$

$$\{E, \vec{J}\} = 0,$$

close a finite-dimensional Lie algebra to which we may apply the GAQ. (Note the minus sign in the first line, which states that the involved group is $SO(3,2)$ and not $SO(4,1)$).

We then quantize the Anti-de Sitter group so that the original operators $\mathcal{H}, \hat{\pi}_i, \hat{\varepsilon}^j$ can be found in its *enveloping algebra* through the expression:

$$\langle \hat{\mathcal{H}} \equiv \frac{1}{8} \hat{E}^2, \quad \hat{\pi}^i \equiv \frac{1}{4} (\hat{g}^{ij} \hat{p}_j + \hat{p}_j \hat{g}^{ij}), \quad \hat{\varepsilon}^i \equiv \frac{1}{2\sqrt{2}} (\hat{\mathcal{H}}^{-1/2} \hat{k}^i + \hat{k}^i \hat{\mathcal{H}}^{-1/2}), \quad \hat{1} \rangle$$

In so doing we parameterize a central extension of the Anti-de Sitter group by (abstract) variables $\{a^0, \vec{a}, \vec{v}, \vec{\epsilon}, \zeta\}$, which mimic those for the Poincaré group. In the same way we hope that the right-invariant generators associated with those parameters reproducing corresponding functions $\{E, p_i, k^j, J^k, 1\}$ as Noether invariants satisfying the Poisson brackets above.

At this point it should be stressed that the parameters $\vec{\epsilon}$ and the corresponding quantum operators (essentially the right generators $\tilde{X}_{\vec{\epsilon}}^R$) are associated with ordinary rotations on Anti-de Sitter space-time, whereas the $\vec{\epsilon}$ parameters correspond to “translations” on the $SU(2)$ manifold.

We shall not give here the explicit group law for the group variables (which can be found in Refs. [33]), limiting ourselves to write the explicit expression for the left-invariant vector fields on the extended $SO(3,2)$ group:

$$\begin{aligned}
\tilde{X}_{a^0}^L &= \frac{1}{q_a} \left\{ \left(\frac{\omega^2}{2c^2} a^0 (\vec{a} \cdot \vec{v}) q_v + \left(1 - \frac{\omega^2}{4c^2} (a^0)^2 \right) k_v \right) \frac{\partial}{\partial a^0} + (2q_v \vec{v} + \frac{\omega^2}{4c^2} (2q_v (\vec{a} \cdot \vec{v}) - a^0 k_v) \vec{a}) \cdot \frac{\partial}{\partial \vec{a}} \right. \\
&\quad \left. - \frac{1}{q_v} \frac{\omega^2}{4c^2} [((2(1 + \vec{v}^2) (\vec{a} \cdot \vec{v}) - 2a^0 q_v) + \vec{a} \cdot \vec{v}) \vec{v} + \vec{a}] \cdot \frac{\partial}{\partial \vec{v}} \right. \\
&\quad \left. + (2q_\epsilon (\vec{a} \times \vec{v}) - \vec{\epsilon} \times (\vec{a} \times \vec{v})) \cdot \frac{\partial}{\partial \vec{\epsilon}} \right\} + \left\{ \frac{1}{q_a} \left(\frac{\omega^2}{2c^2} a^0 (\vec{a} \cdot \vec{v}) q_v + \left(1 - \frac{\omega^2}{4c^2} (a^0)^2 \right) k_v \right) - 1 \right\} \Xi \\
\tilde{X}_{\vec{a}}^L &= \frac{1}{q_a} R(\vec{\epsilon}) \left\{ \left(\frac{\omega^2}{4c^2} a^0 (\vec{a} + 2(\vec{a} \cdot \vec{v}) \vec{v}) + 2q_v \left(1 - \frac{\omega^2}{4c^2} (a^0)^2 \right) \vec{v} \right) \frac{\partial}{\partial a^0} \right. \\
&\quad \left. + \frac{\partial}{\partial \vec{a}} + \left(2 + \frac{\omega^2}{4c^2} (1 + 2\vec{a} \cdot \vec{v}) \right) \vec{v} \vec{v} - \frac{\omega^2}{2c^2} a^0 q_v \vec{v} \vec{a} \right) \cdot \frac{\partial}{\partial \vec{a}} \\
&\quad \left. - \frac{1}{q_v} \frac{\omega^2}{4c^2} [(2q_v (\vec{a} \cdot \vec{v}) - a^0 k_v) \left(\frac{\partial}{\partial \vec{v}} + \vec{v} (\vec{v} \cdot \frac{\partial}{\partial \vec{v}}) \right) \right. \\
&\quad \left. + (-2q_\epsilon q_v \vec{a} + 2q_\epsilon a^0 \vec{v} + q_v (\vec{\epsilon} \times \vec{a}) - a^0 (\vec{\epsilon} \times \vec{v})) \times \frac{\partial}{\partial \vec{\epsilon}} \right] \\
&\quad \left. + \left(\frac{\omega^2}{4c^2} a^0 (\vec{a} + 2(\vec{a} \cdot \vec{v}) \vec{v}) + 2q_v \left(1 - \frac{\omega^2}{4c^2} (a^0)^2 \right) \vec{v} \right) \Xi \right\} \\
\tilde{X}_{\vec{v}}^L &= \frac{1}{q_v} R(\vec{\epsilon}) \left\{ \frac{\partial}{\partial \vec{v}} + \vec{v} \vec{v} \cdot \frac{\partial}{\partial \vec{v}} - (2q_\epsilon \vec{v} - \vec{\epsilon} \times \vec{v}) \times \frac{\partial}{\partial \vec{\epsilon}} \right\} \\
\tilde{X}_{\vec{\epsilon}}^L &= X_{(\vec{\epsilon})}^{R SU(2)} \\
\tilde{X}_{\phi}^L &= i\zeta \frac{\partial}{\partial \zeta} + h.c. \equiv \Xi,
\end{aligned}$$

where:

$$q_a = \sqrt{1 + \frac{\omega^2}{4c^2}(\vec{a}^2 - (a^0)^2)}, \quad q_v = \sqrt{1 + \vec{v}^2}, \quad k_v = 1 + 2\vec{v}^2 \quad \text{and} \quad q_\epsilon = \sqrt{1 - \frac{\vec{\epsilon}^2}{4}}.$$

Following the same steps as those given in the case of the Poincaré group we look for a higher-order polarization leading to the configuration-space “representation”. In fact, the simplest possibility turns out to be generated by a combination of generators formally analogous to that of the Poincaré case, although now the generators are obviously different.

From the polarization condition $\mathcal{D}\psi = 0$ we again arrive at a wave function depending only on a^0 , \vec{a} and satisfying a Klein-Gordon-like equation with $SO(3,2)$ D'Alembertian operator given by

$$\begin{aligned} \square = & \frac{1}{16q_a^2} \left\{ [16 - (a^0)^2 \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial^2}{\partial a^0{}^2} - a^0 \frac{\omega^2}{c^2} [40 + 7 \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2)] \frac{\partial}{\partial a^0} \right. \\ & - [16 + r_a^2 \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial^2}{\partial r_a^2} - \frac{1}{r_a} [32 + \frac{\omega^2}{c^2} (40 + 7 \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial}{\partial r_a} \\ & \left. - 2a^0 r_a \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2)) \frac{\partial^2}{\partial a^0 \partial r_a} \right\} + \frac{\vec{L}^2}{q_a^2 r_a^2}, \end{aligned}$$

where

$$\vec{L}^2 = -\frac{1}{\sin\theta_a} \frac{\partial}{\partial \theta_a} (\sin\theta_a \frac{\partial}{\partial \theta_a}) - \frac{1}{\sin^2\theta_a} \frac{\partial^2}{\partial \varphi_a^2}$$

is the square of the standard orbital angular momentum operator (save for a factor \hbar) and $r_a \equiv \sqrt{\vec{a} \cdot \vec{a}}$.

The *wave functions* are

$$\phi(\vec{a}, a^0) = e^{-2ic\lambda_{nl} \arcsin(\frac{\omega q_a a^0}{\sqrt{4c^2 + \omega^2 q_a^2 r_a^2})} Y_m^l(\theta_a, \varphi_a) (1 + \frac{\omega^2}{c^2} q_a^2 r_a^2)^{-\frac{\lambda_{nl}}{2}} (q_a r_a)^l \phi_l^{\lambda_{nl}}(r_a)$$

where

$$\lambda_{nl} \equiv \frac{E}{\hbar\omega}$$

$$E \equiv \left(\frac{3}{2} + 2n + l + \frac{1}{2} \sqrt{9 + 4 \frac{m^2 c^2}{\hbar^2 \omega^2} - 48\xi} \right) \hbar\omega$$

$$\phi_l^{\lambda_{nl}} = {}_2F_1\left(-n, n + l + \frac{3}{2} - \lambda_{nl}, l + \frac{3}{2}; -\frac{\omega^2}{c^2} q_a^2 r_a^2\right)$$

and ξ is a free parameter related to the “zero-point energy”. On this representation the operators corresponding to the original functions ε^i , π_j and the energy \mathcal{H} can be realized.

Although one can verify by an explicit computation that $\phi(\vec{a}, a^0)$ is actually a solution of the wave equation, a large amount of time can be saved by performing in Ref. [35], eqn. (64)-(66), the change of variables

$$t = \frac{2c}{\omega} \arcsin \left(\frac{\omega q_a a^0}{\sqrt{4c^2 + \omega^2 q_a^2 \vec{a}^2}} \right), \quad \vec{x} = q_a \vec{a},$$

which leads to expressions analogous to ours, although intended to describe the quantum evolution of a free particle moving on an Anti-de Sitter space-time.

As a general comment, we would like to bring the attention of the reader to a potential normal-ordering problem appearing in going from the enveloping algebra of the auxiliary group to the quantum version of the canonical variables in the original Lagrangian formalism. We are referring in particular to the “change” of quantum variables of the form $\hat{x}^j \sim \frac{mc}{2}(\hat{p}_0^{-1} \hat{k}^j + \hat{k}^j \hat{p}_0^{-1})$ (in the case of the free particle). A more general prescription for normal-ordering can be addressed following some sort of “perturbative” group approach to quantization. In fact, it is possible to close order by order in some constants (like $1/c^2$ in the relativistic particle, the structure constants themselves for the sigma model, or coupling constants in general) a Lie algebra which joins together the original variables and those in the auxiliary group. Then, applying the group-quantization technique up to a certain order we arrive at the correct prescription of any operator at the given order.

To conclude we must recognize that at present we are unable to state the class of non-linear systems to which this mechanism can be applied, although much effort is being done in this direction. We will see more examples in next section.

3.4 Phase spaces of $SO(2, 1)$ invariant systems

In this section we shall study the family of 1-D mechanical systems characterized by a Lagrangian of the form:

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \tag{3.13}$$

and Hamiltonian $H = \frac{p^2}{2m} + V(x)$ such that we can find functions $\mathcal{E} = g(H)$, $\mathcal{X} = h(H)f(x)$ and $\mathcal{D} = \{\mathcal{X}, \mathcal{E}\}$ closing a Poisson subalgebra isomorphic to the Lie algebra of $SO(2, 1)$ or $SO(3)$.

We shall see that the function $\mathcal{E} = g(H)$ can always be taken as $\mathcal{E} = \sqrt{\pm H}$, and $h(H)$ will be $h(H) = \sqrt{\pm H}$ or $h(H) = H$.

This transformation will relate an open subset of the phase space of these systems (that with a definite sign of the energy, either positive or negative) with (an open subset of) the coadjoint orbits of the Lie groups $SO(2, 1)$ or $SO(3)$. This transformation is not symplectic and in some cases it fails to be differentiable at some points ($H = 0$), although when restricted to the open subsets of positive or negative energy it is differentiable. This

has important implications at the quantum level, where these systems are realized as certain representations of these groups. We will not carry out the full quantization here. However, some relevant considerations in this respect will be given.

Although there is a large family of potentials fulfilling this property (see for instance [82]), we shall concentrate on the Pöschl-Teller potentials (both trigonometric and hyperbolic) and the Morse Potential since they have many applications in Molecular and Solid State Physics.

The Pöschl-Teller potentials are a family of potentials which can be classified in two types:

- Trigonometric Pöschl-Teller potentials (TPT):

$$V(x) = D \left(\frac{\lambda}{\cos^2(\alpha x)} + \frac{\kappa}{\sin^2(\alpha x)} \right) \quad (3.14)$$

- Hyperbolic (Modified) Pöschl-Teller potentials (MPT)

$$V(x) = -D \left(\frac{\lambda}{\cosh^2(\alpha x)} + \frac{\kappa}{\sinh^2(\alpha x)} \right) \quad (3.15)$$

They have many applications in different branches of Molecular Physics, where they describe out-of-plane bending vibrations, with some variations like the Rosen-Morse Potential [83], and in Solid State Physics, where they model 1-D crystals (Scarf Potential) [84]. The PT potentials also appear in generalizations of the harmonic oscillator potential to spaces of constant curvature [85, 86, 87]. The PT potentials have the same solutions as an harmonic oscillator with position dependent mass, or with energy dependent frequency [88, 6]. They are integrable and related to a $SO(2, 1)$ (or $SO(3)$) dynamical symmetry¹.

Let us study the phase space associated with these systems, in particular the case of the symmetric TPT and MPT potentials.

3.4.1 Trigonometric Pöschl-Teller Potential

We shall concentrate on the Trigonometric PT of the form:

$$V(x) = \frac{D}{\cos^2(\alpha x)} \quad (3.16)$$

which is symmetric and has the form given in Fig.3.1a. The parameter $D > 0$ is the minimum of the potential (the value at the origin), and α is related to the width of the potential.

¹We are referring here to the dynamical group as the one generating the spectrum of the system. This should not be confused with the degeneracy group which span the set of states with the same energy. For instance, in the Hydrogen atom, the groups $SO(2, 2)$, $SO(4)$ and $E(3)$ appear as degeneracy groups for positive, negative and zero energy, respectively.

The trajectories in phase space for the TPT potential can be derived from the equation of motion, either in Lagrangian or Hamiltonian form, and are given by:

$$x(t) = \frac{1}{\alpha} \arcsin \left[\sin(\alpha x_0) \cos(\omega(E)t) + \frac{\alpha}{m\omega(E)} p_0 \cos(\alpha x_0) \sin(\omega(E)t) \right] \quad (3.17)$$

$$p(t) = m\dot{x}(t)$$

where (x_0, p_0) are the initial coordinate and momentum parameterizing each solution and $E = \frac{p_0^2}{2m} + \frac{D}{\cos(\alpha x_0)^2}$ is the energy of this trajectory. In the last expression $\omega(E) \equiv \sqrt{\frac{2\alpha^2 E}{m}}$ is an energy dependent frequency².

These trajectories are shown in Fig.3.1b. If we represent these trajectories in a 3D-phase space (x, p, H) , we obtain the graphic shown in Fig.3.1c. Note that in this case $E \geq D > 0$.

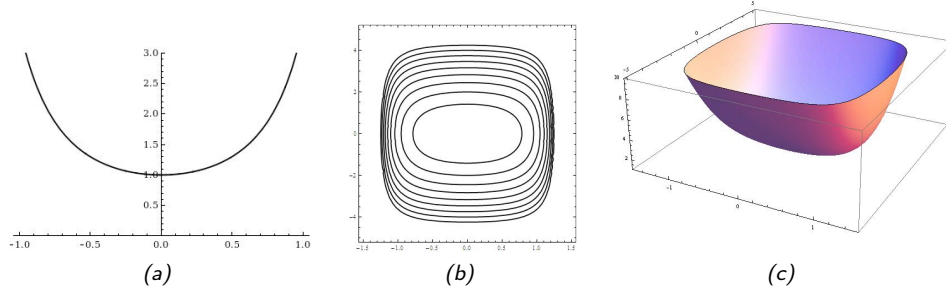


Figure 3.1: Symmetric PT potential (a) and trajectories in 2D (b) and 3D phase space (c).

Closing a subalgebra of the Poisson algebra

We are now interested in closing a Lie algebra with functions of the energy and the position. We start with the basic Poisson bracket, derived from the Poincaré-Cartan 1-form taking the quotient by the solutions of the equation of motion (3.17), which has the canonical form:

$$\{x, p\} = 1, \quad (3.18)$$

with $p = \frac{\partial L}{\partial \dot{x}}$. Although $\{H, x, p\}$ do not close a Poisson subalgebra, we can find the functions closing the algebra $SO(2, 1)$:

$$\mathcal{E} \equiv 2\sqrt{D}\sqrt{H}, \quad \mathcal{X} \equiv \frac{2}{\sqrt{m}\Omega} \sqrt{H} \sin(\alpha x), \quad \mathcal{P} \equiv \sqrt{2} p \cos(\alpha x), \quad (3.19)$$

with $\Omega = \omega(D)$, i.e. the frequency at the bottom of the potential, corresponding to the limit of small oscillations.

²The solutions of the classical equation of motion correspond to that of an harmonic oscillator with energy dependent frequency (see [6] and also [89]).

In fact, we find:

$$\{\mathcal{E}, \mathcal{P}\} = m\Omega^2 \mathcal{X}, \quad \{\mathcal{E}, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} \mathcal{E}. \quad (3.20)$$

This is an $SO(2, 1)$ algebra where the compact generator is \mathcal{E} and the Casimir is:

$$\frac{1}{D} \mathcal{E}^2 - m\Omega^2 \mathcal{X}^2 - \frac{1}{m} \mathcal{P}^2 = 4D > 0. \quad (3.21)$$

The coadjoint orbit of $SO(2, 1)$ associated with this system is the upper sheet of a two-sheet hyperboloid, since the Casimir is positive and $\mathcal{E} > 0$.

The Quantum TPT oscillator

It should be stressed that the transformation (3.19) is not symplectic since $\{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} \mathcal{E} \neq 1$. This is not a drawback, however, and the dynamics (both classical and quantum) can be derived entirely from $SO(2, 1)$. In this case the transformation (3.19) is differentiable, and the phase space shown in Fig.3.1c is mapped to a full coadjoint orbit of $SO(2, 1)$. This implies that at the quantum level this system will be realized as a unitary irreducible representation of the discrete series of representations of $SO(2, 1)$ (see, for instance, [90]), the one associated with the upper sheet of a two-sheet hyperboloid in the sense of the Coadjoint Orbit Method of Kirillov [10] (see also [25]).

3.4.2 Modified Pöschl-Teller Potential

We shall concentrate on the MPT of the form:

$$V(x) = -\frac{D}{\cosh^2(\alpha x)} \quad (3.22)$$

which is also symmetric and has the form given in Fig.3.2a. The parameter $D > 0$ is the potential depth and α is related to the width of the potential.

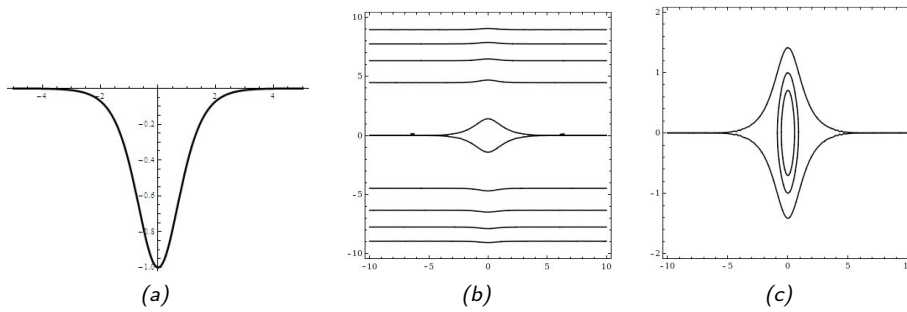


Figure 3.2: Symmetric MPT potential (a) and trajectories in 2D phase space for $E > 0$ (b) and $E < 0$ (c).

The trajectories in phase space for the MPT potential are given by:

$$x(t) = \frac{1}{\alpha} \operatorname{arcsinh} \left[\sinh(\alpha x_0) \cosh(\omega(E)t) + \frac{\alpha}{m\omega(E)} p_0 \cosh(\alpha x_0) \sinh(\omega(E)t) \right] \quad (3.23)$$

$$p(t) = m\dot{x}(t)$$

where (x_0, p_0) are the initial coordinate and momentum parameterizing each solution and $E = \frac{p_0^2}{2m} - \frac{D}{\cosh^2(ax_0)}$ is the energy of this trajectory. Here $\omega(E)$ has the same meaning as in the TPT case. Note that when $E < 0$ the time-dependent hyperbolic functions change to their trigonometric counterparts. These trajectories are shown in Fig.3.2b for $E > 0$ and Fig.3.2c for $E < 0$. For $E = 0$ they degenerate to $x(t) = \frac{1}{a} \operatorname{arcsinh} \left[\sinh(ax_0) + \cosh(ax_0) \frac{\alpha p_0}{m} t \right]$ which is a free motion in the coordinate $\xi = \sinh(ax)$, $\xi = \xi_0 + \dot{\xi}_0 t$ with constant velocity $\dot{\xi}_0 = \frac{\alpha}{m} p_0 \sqrt{1 + \xi_0^2}$, and p_0 related to x_0 through the constraint $E = 0$.

The trajectories in a 3D-phase space (x, p, H) are show in Fig.3.3a for $E > 0$, Fig.3.3b for $E < 0$ and Fig.3.3c for the union of both cases.

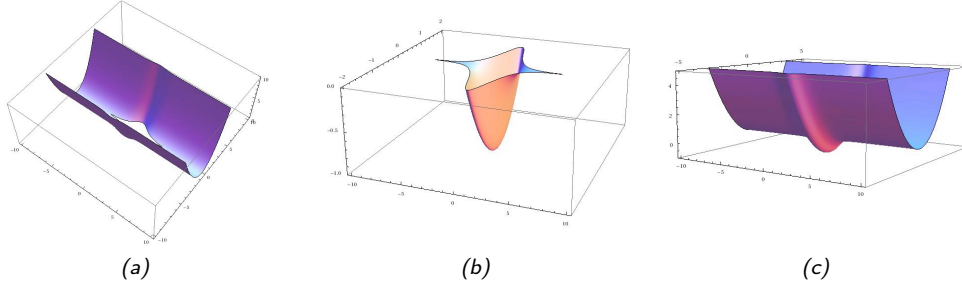


Figure 3.3: Trajectories in 3D phase space for $E > 0$ (a), $E < 0$ (b) and the union of both cases (c), for the MPT Potential

Closing a subalgebra of the Poisson algebra

The basic Poisson bracket, derived as in the case of the TPT, has the canonical form:

$$\{x, p\} = 1. \quad (3.24)$$

Although $\{H, x, p\}$ do not close a Poisson subalgebra, we can find functions closing an algebra, but we must distinguish between the cases $E > 0$ and $E < 0$.

- $E > 0$

The following functions close a $SO(2, 1)$ algebra:

$$\mathcal{E} \equiv 2\sqrt{D}\sqrt{H}, \quad \mathcal{X} \equiv \frac{2}{\sqrt{m}\Omega} \sqrt{H} \sinh(ax), \quad \mathcal{P} \equiv \sqrt{2}p \cosh(ax). \quad (3.25)$$

In fact, we find:

$$\{\mathcal{E}, \mathcal{P}\} = -m\Omega^2 \mathcal{X}, \quad \{\mathcal{E}, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} \mathcal{E}, \quad (3.26)$$

where again $\Omega = \omega(D)$. This is a $SO(2, 1)$ algebra where the compact generator is \mathcal{P} and the Casimir is:

$$\frac{1}{m} \mathcal{P}^2 - \frac{1}{D} \mathcal{E}^2 - m\Omega^2 \mathcal{X}^2 = 4D > 0. \quad (3.27)$$

The coadjoint orbit of $SO(2, 1)$ associated with this system is a half of a two-sheet hyperboloid, shown in Fig.3.4a.

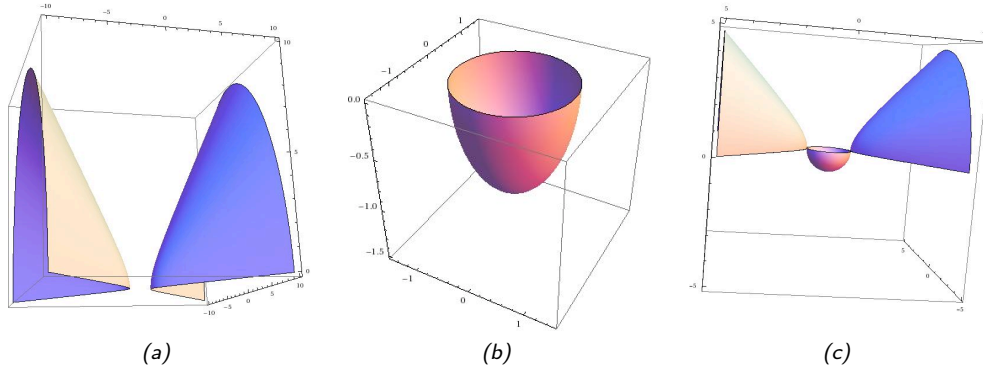


Figure 3.4: Coadjoint orbits of the group $SO(2,1)$ for $E > 0$ (a), $SO(3)$ for $E < 0$ (b) and the union of both cases (c).

- $E < 0$

The following functions close a $SO(3)$ algebra³:

$$\mathcal{E} \equiv -2\sqrt{D}\sqrt{-H}, \quad \mathcal{X} \equiv \frac{2}{\sqrt{m\Omega}}\sqrt{-H}\sinh(\alpha x), \quad \mathcal{P} \equiv \sqrt{2}p \cosh(\alpha x). \quad (3.28)$$

In fact, we find:

$$\{\mathcal{E}, \mathcal{P}\} = m\Omega^2 \mathcal{X}, \quad \{\mathcal{E}, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} \mathcal{E}. \quad (3.29)$$

This is an $SO(3)$ algebra with the Casimir function given by:

$$\frac{1}{D} \mathcal{E}^2 + m\Omega^2 \mathcal{X}^2 + \frac{1}{m} \mathcal{P}^2 = 4D > 0. \quad (3.30)$$

The coadjoint orbit of $SO(3)$ associated with this system is a hemisphere, shown in Fig.3.4b.

Putting the two orbits together we obtain a phase space, shown in Fig.3.4c, resembling that of Fig.3.3c.

As in the TPT case, the transformations (3.25) and (3.28) are not symplectic. In this case they are not even differentiable at $H = 0$. As shown in Fig.3.4c, the whole $E = 0$ closet subset of phase space in (x, p, H) is mapped to two points in $(\mathcal{X}, \mathcal{P}, \mathcal{E})$, namely $(0, \pm 2\sqrt{mD}, 0)$. This means that the two cases, $E > 0$ and $E < 0$ are disconnected, and there is no way to connect them in the framework of the groups $SO(2, 1)$ and $SO(3)$.

The Quantum MPT Oscillator

At the quantum level, this system has special features, since we must distinguish between the cases $E > 0$ and $E < 0$, and in neither cases the phase space is a coadjoint orbit, but a half of it.

³Note the minus sign in \mathcal{E} in order to have the same sign as H .

- $E > 0$

The system will be realized as a part of the sum of two unitary and irreducible representations of the discrete series (positive and negative) of (the universal covering group of) $SO(2, 1)$. The Hamiltonian is a non-compact operator with positive continuum spectrum. This result seems to be in agreement with [91], where the discrete series of $SO(2, 1)$ is used to describe the continuum spectrum of the MPT potential.

- $E < 0$

The system will be realized as a part of an irreducible representation of $SU(2)$. The Hamiltonian is a compact operator with discrete spectrum, but only one half of the $2s + 1$ states of the $SU(2)$ representations are realized.

In the last case, there is a better interpretation as a non-unitary, finite-dimensional representation of (the universal covering group of) $SO(2, 1)$ (see [6]). The reason is that in the context of $SU(2)$ there is no explanation for the fact that half of the states are missing, and that the potential depth should have only definite values, since it is related to the values of the Casimir, which in turns depends on the discrete values of the spin.

Here $SO(3)$ appears as contained in the complexification of $SO(2, 1)$, but then it should be realized in a non-unitary way. But $SO(2, 1)$ admits non-unitary, finite dimensional representations, which in many respects behave as those of $SU(2)$. The lack of unitarity manifest itself in that half of the states are non-normalizable, and therefore are outside of the Hilbert space of the physical (normalizable) states. These non-normalizable states are precisely the antibound states (ABS), which represent outgoing states growing at $\pm\infty$ (see [92]).

3.4.3 The Morse Potential

The Morse Potential is given by:

$$V(x) = D(e^{-2ax} - 2e^{-ax}) \quad (3.31)$$

and is shown in Fig.3.5a.

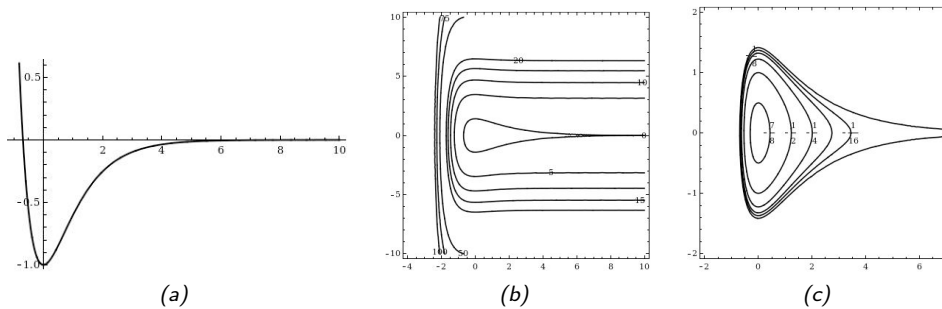


Figure 3.5: Morse potential (a) and trajectories in 2D phase space for $E > 0$ (b) and $E < 0$ (c).

The trajectories in phase space for the Morse potential can be obtained by solving the equations of motion, and are given by:

$$x(t) = \frac{1}{\alpha} \log \left[-\frac{D}{E} + \left(e^{\alpha x_0} + \frac{D}{E} \right) \cosh(\omega(E)t) + \frac{\alpha}{m\omega(E)} p_0 e^{\alpha x_0} \sinh(\omega(E)t) \right] \quad (3.32)$$

$$p(t) = m\dot{x}(t)$$

where (x_0, p_0) are the initial coordinate and momentum parameterizing each solution and $E = \frac{p_0^2}{2m} + D(e^{-2\alpha x_0} - 2e^{-\alpha x_0})$ is the energy of this trajectory. Here $\omega(E)$ has the same meaning as in the MPT and TPT case. Note that when $E < 0$ the time-dependent hyperbolic functions change to their trigonometric counterparts. These trajectories are shown in Fig.3.5b for $E > 0$ and Fig.3.5c for $E < 0$. The trajectories in a 3D-phase space (x, p, H) are shown in Fig.3.6a for $E > 0$, Fig.3.6b for $E < 0$ and Fig.3.6c for the union of both cases.

For $E = 0$ they degenerate to $x(t) = \frac{1}{\alpha} \log \left[e^{\alpha x_0} + e^{\alpha x_0} \frac{\alpha p_0}{m} t \right]$ which is a free motion in the coordinate $\xi = e^{\alpha x} + \frac{D}{E}$, $\xi = \xi_0 + \dot{\xi}_0 t$ with constant velocity $\dot{\xi}_0 = \frac{\alpha}{m} p_0 e^{\alpha x_0}$, and p_0 related to x_0 through the constraint $E = 0$.

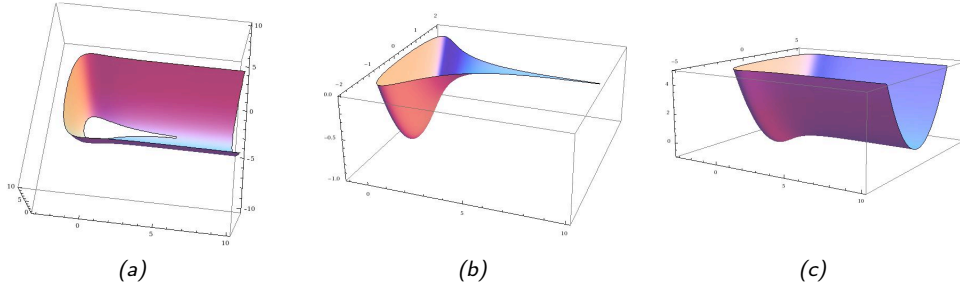


Figure 3.6: Trajectories in 3D phase space for $E > 0$ (a), $E < 0$ (b) and the union of both cases (c) for the Morse Potential.

Closing a subalgebra of the Poisson algebra

The basic Poisson bracket, derived as in the previous cases, has the canonical form:

$$\{x, p\} = 1. \quad (3.33)$$

Although $\{H, x, p\}$ do not close a Poisson subalgebra, we can find again functions closing an algebra, but we must also distinguish between the cases $E > 0$ and $E < 0$.

- $E > 0$

The following functions close a $SO(2, 1)$ algebra:

$$\mathcal{E} \equiv 2\sqrt{D}\sqrt{H}, \quad \mathcal{X} \equiv \frac{\sqrt{2}}{\alpha D} H e^{\alpha x} + \frac{\sqrt{2}}{\alpha}, \quad \mathcal{P} \equiv \frac{\sqrt{2H}}{\sqrt{D}} e^{\alpha x} p. \quad (3.34)$$

In fact, we find:

$$\{\mathcal{E}, \mathcal{P}\} = -m\Omega^2 \mathcal{X}, \quad \{\mathcal{E}, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = -\frac{1}{D} \mathcal{E}. \quad (3.35)$$

This is a $SO(2, 1)$ algebra where the compact generator is \mathcal{X} . The Casimir is:

$$m\Omega^2 \mathcal{X}^2 - \frac{1}{D} \mathcal{E}^2 - \frac{1}{m} \mathcal{P}^2 = 4D > 0. \quad (3.36)$$

The coadjoint orbit of $SO(2, 1)$ associated with this system is a half of the upper sheet of a two-sheet hyperboloid, shown in Fig.3.7a

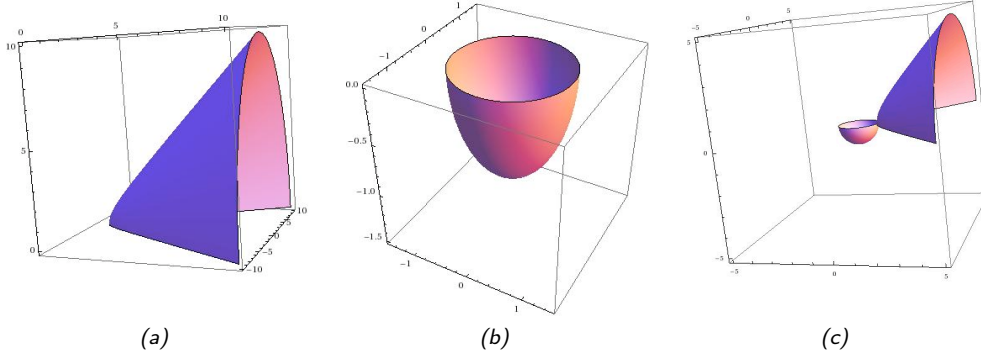


Figure 3.7: Coadjoint orbits of the group $SO(2,1)$ for $E > 0$ (a), $SO(3)$ for $E < 0$ (b) and the union of both cases (c), for the Morse Potential

- $E < 0$

The following functions close a $SO(3)$ algebra:

$$\mathcal{E} \equiv -2\sqrt{D}\sqrt{-H}, \quad \mathcal{X} \equiv \frac{\sqrt{2}}{\alpha D} H e^{\alpha x} + \frac{\sqrt{2}}{\alpha}, \quad \mathcal{P} \equiv \frac{\sqrt{-2H}}{\sqrt{D}} e^{\alpha x} p. \quad (3.37)$$

In fact, we find:

$$\begin{aligned} \{\mathcal{E}, \mathcal{P}\} &= -m\Omega^2 \mathcal{X} \\ \{\mathcal{E}, \mathcal{X}\} &= \frac{1}{m} \mathcal{P} \\ \{\mathcal{X}, \mathcal{P}\} &= \frac{1}{D} \mathcal{E}. \end{aligned} \quad (3.38)$$

This is a $SO(3)$ algebra, with the Casimir given by:

$$\frac{1}{D} \mathcal{E}^2 + m\Omega^2 \mathcal{X}^2 + \frac{1}{m} \mathcal{P}^2 = 4D > 0. \quad (3.39)$$

The coadjoint orbit of $SO(3)$ associated with this system is a hemisphere, shown in Fig.3.7b.

Putting the two orbits together we obtain a phase space, shown in Fig.3.7c that resembles that of Fig.3.6c.

As in the TPT and MPT cases, the transformation (3.34) and (3.37) is not symplectic, and as in the MPT case, it is not differentiable at $H = 0$. The same comments as in the MPT case apply here, with the difference that now the $E = 0$ states are mapped to the single point $(\frac{\sqrt{2}}{a}, 0, 0)$.

The Quantum Morse Oscillator

At the quantum level, this system has special features, since we must distinguish between the cases $E > 0$ and $E < 0$, and in neither cases the phase space is a coadjoint orbit, but a half of it.

- $E > 0$

The system will be realized as a part of a unitary and irreducible representations of the positive discrete series of (the universal covering group of) $SO(2, 1)$. The Hamiltonian is a non-compact operator with positive continuum spectrum.

- $E < 0$

This system will be realized as a part of an irreducible representation of $SO(3)$. The Hamiltonian is a compact operator with discrete spectrum, but only one half of the $2s + 1$ states of the $SU(2)$ representations are realized.

In this last case, there is a better interpretation as a non-unitary, finite-dimensional representation of (the universal covering group of) $SO(2, 1)$, see [93]. The same comments as in the MPT case applies here, the lacking states can be interpreted as non-normalizable antibound states (see [92]).

Comments

In this section a transformation of (an open subset of) the phase space of some $SO(2, 1)$ invariant systems into an open subset of a coadjoint orbit of $SO(2, 1)$ or $SO(3)$ is given. For the case of the TPT potential, the transformation maps the whole phase space into a whole coadjoint orbit of $SO(2, 1)$, and therefore the group-theoretical interpretation of this case seems rather clear. For the MPT and Morse potentials, we must distinguish between positive and negative energies, and in both cases each open subset of phase space is mapped into an open subset of a coadjoint orbit, which in addition belong to different Lie groups, $SO(2, 1)$ for positive energy and $SO(3)$ for negative energy.

This behaviour is rather unpleasant, and a unified description, into a single Lie group, would be desirable. Probably this would require an infinite-dimensional group able to accommodate the Hamiltonian, which has a mixed spectrum (both positive continuum and negative discrete spectra), as a single generator. However, an intermediate step can be done including both groups into a finite-dimensional group, the price we must pay being that the Hamiltonian is associated with different generators for the positive and negative energy cases. The minimal algebra including both $SO(2, 1)$ and $SO(3)$ algebras is the complexification of any of them, the $SL(2, \mathbb{C})$ algebra, which is isomorphic to $SO(3, 1)$, the

Lorentz algebra. The complexification appears here in a natural way due to the presence of the square root of the Hamiltonian in the transformations (3.25) and (3.34). Thus, $SL(2, \mathbb{C})$ is the natural framework to study this problem from a group-theoretical point of view (this was already anticipated in [6]).

For instance, if in the MPT potential for positive energy, besides the functions (3.25) we introduce $\mathcal{E}' \equiv i\mathcal{E}$, $\mathcal{X}' \equiv i\mathcal{X}$ and $\mathcal{P}' \equiv i\mathcal{P}$, the set $\{\mathcal{E}, \mathcal{X}, \mathcal{P}, \mathcal{E}', \mathcal{X}', \mathcal{P}'\}$ closes a $SL(2, \mathbb{C})$ algebra where the $SO(2, 1)$ subalgebra is the original $\{\mathcal{E}, \mathcal{X}, \mathcal{P}\}$ and the $SO(3)$ subalgebra is $\{\mathcal{E}', \mathcal{X}', \mathcal{P}'\}$, which coincides with the one given in (3.28) and that are real for negative energies.

A similar construction can be done for the Morse Potential case. What remains to be done is to relate the phase spaces of these systems to the coadjoint orbits (or open subsets of them) of $SL(2, \mathbb{C})$, and to derive Hilbert space of quantum states from the unitary (or non-unitary) representations of $SL(2, \mathbb{C})$. This is work in progress.

Finally, we would like to comment on the appearance of $\mathcal{E} = \sqrt{\pm H}$ in all the transformations studied in this Section. The reason for this is that, in our approach, both the classical and quantum equation of motions of the systems, which are non-relativistic and therefore are linear in the Hamiltonian, are obtained through the Casimir function or operator of semisimple Lie algebras, which are quadratic in all generators (they could be seen relativistic Klein-Gordon equations of motion, as opposed to the Schrödinger equation which is linear in the Hamiltonian).

In fact, when we apply our procedure to a relativistic system like a free relativistic particle, with a quadratic equation of motion, a similar construction can be done but in this case the transformation does not contain \sqrt{H} . In fact $\mathcal{E} = H$ and $h(H) = H$, leading to the Poincaré algebra (see Section 3.2 for details), that it is not a semisimple algebra but its Casimir is still quadratic.

3.5 Particle on a sphere \mathbb{S}^2

The quantization of simple second-class constrained mechanical systems has usually been accomplished by adopting as canonical commutation relations those given by the so-called Dirac algorithm. This method, although successful in many situations, can potentially lead to wrong results, and additional strategies or physical considerations are needed. This, which is not a serious problem when the system under consideration contains “known physics”, as in the case of the free particle constrained to move on a sphere surface [67, 68, 69], could become dramatic if we want to unfold new physical features.

In general, canonical quantization makes extensive use of Hamiltonian formulation of classical mechanics, trying to put the classical theory in a form in which the quantization would be straightforward and/or as unambiguous as possible. However, it should be recalled once again that it is the quantum system what contains the physical entity, the classical theory being a mere approximation.

This Section is devoted to show how, in the case of the partial-trace non-linear sigma model (NLSM) at least, a proper quantum theory can be obtained straightforwardly if one leaves aside the idea that the classical theory must lead to the quantum one through the inherently ambiguous process of *quantization*. Here we suggest that the *quantized system* should be *directly* obtained through a deeper knowledge of the corresponding dynamical

symmetry transformations, i.e. those that are able to span the whole space of physical states (and/or parameterize the classical solution manifold by means of Noether invariants). We aim at obtaining the quantum theory not from an appropriate treatment of canonical classical quantities, but from the clear algorithm GAQ based on the complete symmetry of the system, which naturally selects the basic quantum operators.

After introducing the general setting of the problem we shall be explicitly concerned with the example of $SU(2)$.

3.5.1 Symmetries

We shall restrict ourselves to the quantum mechanical case $D = 1$, so that the fields $\epsilon^i(x^\mu)$ are just curves $\epsilon^i(t)$ on Σ , which we shall take as a (semisimple, linear) Lie group manifold G , or a given coset G/G_λ (see later). Let us denote by $g = e^{\epsilon^i T_i}$ an element of G , where $T_i, i = 1, \dots, \dim(G)$, stands for the Lie algebra generators, with commutation relations $[T_i, T_j] = C_{ij}^k T_k$. For the sake of simplicity, we shall choose T_i in the adjoint representation, whose matrix elements are $(T_i)_j^k = C_{ij}^k$. Then the Killing form, used to raise and lower indices, is given by $K_{ij} \equiv C_{il}^k C_{jk}^l = \text{Tr}_G(T_i T_j)$. With this notation, the NLSM Lagrangian acquires the simple algebraic form:

$$\mathcal{L}_G = \frac{1}{2} \kappa \text{Tr}_G(\theta \theta), \quad (3.40)$$

where $\theta \equiv g^{-1} \dot{g}$ and here κ is intended to have the dimensions of an inertia moment.

It is well-known the difficulty found by canonical quantization in dealing with non-linear systems. Even symmetry-based quantization techniques face the impossibility of parameterizing the solution manifold by a finite dimensional Lie group. This is essentially because $g^{-1} \dot{g}$ is not a total derivative, except for Abelian groups. However, this obstruction disappears when the manifold Σ is considered to be a coset G/G_λ of G , G_λ being the isotropy subgroup of a given Lie algebra element λ under the adjoint action $\lambda \rightarrow g \lambda g^{-1}$. To be precise λ should have been defined as an element of the dual of the Lie algebra, which is equivalent to the Lie algebra since G is semisimple. In this sense λ , which can be seen as an ordinary vector, may be endowed with length dimensions so that κ would appear as a mass m rather than an inertia moment.

In this case, the (total-trace) NLSM Lagrangian (3.40) takes the (partial-trace) form:

$$\mathcal{L}_{G/G_\lambda} = \frac{1}{2} m \text{Tr}_{G/G_\lambda}(\theta \theta) \equiv \frac{1}{2} m \text{Tr}_G(\theta_\lambda \theta_\lambda), \quad (3.41)$$

where we have defined

$$\theta_\lambda \equiv [\theta, \lambda].$$

It can be realized that, defining

$$S \equiv g \lambda g^{-1}, \quad g \in G,$$

we have an alternative way of writing (4.14) as

$$\mathcal{L}_{G/G_\lambda} = \frac{1}{2} m \text{Tr}_G(\dot{S} \dot{S}) = \frac{1}{2} m K_{ij} \dot{S}^i \dot{S}^j, \quad (3.42)$$

where $S^i \equiv \text{Tr}_G(T^i S)$. Note that this Lagrangian is singular due to the existence of constraints like, for example, $\text{Tr}_G(S^2) = \text{Tr}_G(\lambda^2) \equiv r^2$. We shall not deal with constraints at this stage. They will be naturally addressed inside our quantization procedure below.

We can try to find two sets of generators mimicking the basic symmetry of the Galilean particle (i.e., translations and boosts). For the first set, we choose the generators of the group itself:

$$X_i = C_{ij}{}^k S^j \frac{\partial}{\partial S^k} + C_{ij}{}^k \dot{S}^j \frac{\partial}{\partial \dot{S}^k}. \quad (3.43)$$

The Lagrangian (3.42) is *strictly* invariant under the action of these generators, i.e.:

$$L_{X_i} \mathcal{L}_{G/G_\lambda} = 0, \quad \forall i = 1, \dots, \dim(G),$$

where L_X stands for the Lie derivative with respect to a generator X . In this computation, the fact that the product $K_{lm} C_{ij}{}^l \equiv C_{ijm}$ is fully antisymmetric has been used.

As far as the second set of symmetries is concerned (those playing the role of boosts), we propose the following one:

$$X'_i = \frac{1}{r^2} C_{ij}{}^k C_{lk}{}^n S^j S^l \frac{\partial}{\partial \dot{S}^n}. \quad (3.44)$$

These generators leave the Lagrangian *semi*-invariant in the sense that they give a total derivative (thus leaving the action strictly invariant), much in the same way the generators of boosts do in the free Galilean particle, that is,

$$L_{X'_i} \mathcal{L}_{G/G_\lambda} = m \dot{S}_i, \quad \forall i = 1, \dots, \dim(G). \quad (3.45)$$

The generators (3.43) and (3.44) close a finite-dimensional Lie algebra with commutation relations

$$[X_i, X_j] = -C_{ij}{}^k X_k, \quad [X_i, X'_j] = -C_{ij}{}^k X'_k, \quad [X'_i, X'_j] = 0. \quad (3.46)$$

The corresponding symmetry group is the (co-)tangent group of G and will be denoted by $G^{(1)}$ (see [70] as regards gauge theory).

We shall assume this algebra (in fact, a central extension of it) as the basic symmetry of the quantum particle constrained to move on the manifold G/G_λ . In fact, the semi-invariance (4.11) of the Lagrangian suggests the presence of a central extension $\tilde{G}^{(1)}$ of the group $G^{(1)}$, as happens in the quantum Galilean particle [71]. At the Lie algebra level, this central extension only affects the second commutator in (3.46), which now reads

$$[\tilde{X}_i, \tilde{X}'_j] = -C_{ij}{}^k \tilde{X}'_k - C_{ij}{}^k \lambda_k \frac{m}{\hbar} \Xi \quad (3.47)$$

where Ξ denotes the central generator.

This centrally extended group $\tilde{G}^{(1)}$ is the group of *strict* invariance of the system, and contains the necessary information to obtain the quantum theory. Although this central extension is trivial from a strict mathematical point of view, in the sense that a redefinition of a generator eliminates the central generator from the r.h.s. of the Lie algebra commutators, physically it behaves as a non-trivial one, since under an Inönü-Wigner contraction

(limit process) leads to a non-trivial extension of the contracted group [72] (see also [73] and references therein).

It should be remarked that the commutation relations providing the central term on the right hand side generalize those of the Heisenberg-Weyl algebra, where \tilde{X}_i play the role of (non-Abelian) “translation” generators, and \tilde{X}'_j the role of “boost” generators, although restricted to the coset space G/G_λ . This point will be further clarified in the example of the $SU(2)$ group.

3.5.2 Quantum Theory

The leading idea is to obtain an irreducible and unitary representation (unirep) of the basic group suggested in the previous section, i.e. the (centrally extended) $\tilde{G}^{(1)}$. The central extension will select a specific representation, associated with a given coadjoint orbit.

The procedure presented above is quite simple when applied to the case of a particle moving on the coset space $\mathbb{S}^2 \equiv SU(2)/U(1)$. We begin with a specific realization of the basic group $\tilde{G}^{(1)}$, which in this case turns out to be a centrally extended Euclidean Group. In it, the fields X_{e^i} , $i = 1, 2, 3$ generate ordinary rotations, parameterized by vectors $\vec{\epsilon}$ whose direction determines the axis of rotation and its modulus the angle of rotation χ by $|\vec{\epsilon}| = 2 \sin \frac{\chi}{2}$. In the same way, the fields X_{θ^i} , $i = 1, 2, 3$ correspond to the (co-)tangent subgroup, parameterized by θ^i . In terms of these variables the group law is:

$$R(\vec{\epsilon}'') = R(\vec{\epsilon}')R(\vec{\epsilon})$$

$$\vec{\theta}'' = R^{-1}(\vec{\epsilon})\vec{\theta}' + \vec{\theta}$$

$$\zeta'' = \zeta'\zeta \exp \left\{ i \frac{mr}{\hbar} \vec{\lambda} \cdot (\vec{\theta}'' - \vec{\theta}' - \vec{\theta}) \right\} = \zeta'\zeta \exp \left\{ i \frac{mr}{\hbar} \vec{\lambda} \cdot (R^{-1}(\vec{\epsilon})\vec{\theta}' - \vec{\theta}') \right\}$$

$$\left(R(\vec{\epsilon})^i_j \equiv \left(1 - \frac{\vec{\epsilon}^2}{2} \right) \delta^i_j - \sqrt{1 - \frac{\vec{\epsilon}^2}{4}} \eta^i_{jk} \epsilon^k + \frac{1}{2} \epsilon^i \epsilon_j \right),$$

where $\vec{\lambda}$ is an arbitrary, constant vector in the (co-)algebra with modulus r , and $\zeta \equiv e^{i\phi}$ is the quantum mechanical phase. The \hbar constant has been introduced to keep the exponent dimensionless.

We can immediately calculate the corresponding infinitesimal generators of the left action, which are the right-invariant vector fields:

$$\tilde{X}_{\vec{\epsilon}}^R = X_{\vec{\epsilon}}^{R(SU(2))}$$

$$\tilde{X}_{\vec{\theta}}^R = R^{-1} \frac{\partial}{\partial \vec{\theta}} + \frac{mr}{\hbar} (R\vec{\lambda} - \vec{\lambda})^t \Xi$$

$$\tilde{X}_{\phi}^R = \text{Re} \left(i \zeta \frac{\partial}{\partial \zeta} \right) \equiv \Xi.$$

Needless to say that this set of generators closes (a particular case of) the Lie algebra that we found in the previous section:

$$[\tilde{X}_{\epsilon^i}^R, \tilde{X}_{\epsilon^j}^R] = -\eta_{ij}{}^k \tilde{X}_{\epsilon^k}^R$$

$$[\tilde{X}_{\epsilon^i}^R, \tilde{X}_{\theta^j}^R] = -\eta_{ij}{}^k \left(\tilde{X}_{\theta^k}^R + \frac{mr}{\hbar} \lambda_k \Xi \right)$$

$$[\tilde{X}_{\theta^i}^R, \tilde{X}_{\theta^j}^R] = 0.$$

We can compute as well the infinitesimal generators of the right action, which are the left-invariant vector fields:

$$\tilde{X}_{\epsilon}^L = X_{\epsilon}^{L(SU(2))} - \vec{\theta} \wedge \frac{\partial}{\partial \vec{\theta}} - \frac{mr}{\hbar} \vec{\theta} \wedge \vec{\lambda} \Xi$$

$$\tilde{X}_{\theta}^L = \frac{\partial}{\partial \vec{\theta}}$$

$$\tilde{X}_{\phi}^L = \text{Re} \left(i \zeta \frac{\partial}{\partial \zeta} \right) \equiv \Xi,$$

closing the same Lie algebra but with opposite structure constants.

The characteristic module, i.e., the sub-algebra generated by those vector fields which do not produce a central term under commutation, and therefore without dynamical content, is generated by two fields:

$$\mathcal{G}_{\Theta} = \langle \vec{\lambda} \cdot \tilde{X}_{\epsilon}^L, \vec{\lambda} \cdot \tilde{X}_{\theta}^L \rangle,$$

A polarization sub-algebra is given by the characteristic module together with half of the conjugated pairs:

$$\mathcal{P} = \langle \vec{\lambda} \cdot \tilde{X}_{\epsilon}^L, \tilde{X}_{\theta}^L \rangle.$$

This must not be interpreted as constraint conditions, since they preserve the action of the right-invariant vector fields, i.e., those which will be the physical operators. An alternative treatment, making explicit use of constraints can be seen, for instance, in [74].

An irreducible representation of the group is given by the action of the right-invariant vector fields of the group on the complex functions valued over the group manifold, provided that these functions are polarized and satisfy the condition of U(1)-function:

$$\mathcal{P}\Psi = 0, \quad \Xi\Psi = i\Psi.$$

It can be easily checked that such functions (now true *wave functions*) are of the form

$$\Psi \left(R(\vec{\epsilon}) \vec{\lambda}, \vec{\theta}, \zeta \right) = \zeta \Phi \left(R(\vec{\epsilon}) \vec{\lambda} \right),$$

where Φ is an arbitrary function, or equivalently, in terms of the variable $\vec{S} \equiv R(\vec{e})\vec{\lambda}$ (the positions on the surface of a sphere of radius r):

$$\Psi(\vec{S}, \vec{\theta}, \zeta) = \zeta \Phi(\vec{S}).$$

The explicit action of the right-invariant vector fields on these wave functions is computed to give:

$$\tilde{X}_{\vec{e}}^R \Psi = \zeta (\vec{S} \wedge \vec{\nabla} \Phi) = \vec{S} \wedge \vec{\nabla}_{\vec{S}} \Psi$$

$$\tilde{X}_{\vec{\theta}}^R \Psi = i \frac{mr}{\hbar} (\vec{S} - \vec{\lambda}) \Psi.$$

It is possible to redefine the vector fields to obtain the actual quantum operators, acting only on the arbitrary part of the wave functions. Thus, we end up with the explicit representation over the wave-functions Φ depending only in the variable $\vec{S} = R\vec{\lambda}$,

$$\hat{L}\Phi(\vec{S}) \equiv i\hbar \vec{S} \wedge \vec{\nabla}\Phi(\vec{S})$$

$$\hat{S}\Phi(\vec{S}) \equiv \vec{S}\Phi(\vec{S}).$$

It becomes evident at this point that the domain of the wave functions have been naturally selected without imposing any constraint condition as such.

As was previously emphasized, the \hat{L} operators now play the role of “generators of translations” on the surface of the sphere, \hat{S} playing that of a “position operator” on the sphere surface⁴.

Finally, to obtain a Hamiltonian, we proceed as in the free particle: it is defined as the generator of “translations” squared, so that

$$\hat{H} = \frac{\hat{L}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2.$$

There is no ambiguity in this expression, since it corresponds to the squared action of \hat{L} as a “basic operator”. \hat{H} thus provides the energy spectrum

$$E_l = \frac{\hbar^2}{2m} l(l+1)$$

with no extra terms, in contrast with standard Dirac quantization, as remarked in [75]. Note that \hat{H} coincides with the Casimir of $SU(2)$ restricted to \mathbb{S}^2 . In the general case, the energy operator will be the quadratic Casimir of G restricted to G/G_λ , and this turns out to be the Laplace-Beltrami operator on G/G_λ .

⁴Note that, in this representation of the Euclidean Group, the relation $\hat{S} \cdot \hat{L} = 0$ is fulfilled. However, the Euclidean Group admits another family of central extensions, where $\hat{S} \cdot \hat{L} \neq 0$, and the corresponding representations can realize a magnetic monopole in the center of the sphere.

Massive Yang-Mills Theories

In the 1960's the mechanism of spontaneously broken symmetry, usually referred to as the Higgs-Kibble mechanism [94], came into the particle physics scenario [95], imported from solid state physics (mainly in relation to Meissner effect), to match the masses of the intermediate vector bosons with renormalizability [96]. However, in spite of the wide acceptance today of the Standard Model of electroweak interactions as a whole and of its phenomenological accuracy (putting aside the existence of the Higgs particle), there exists a rather extended feeling that a deeper structure is underneath, owing specially to the artificiality of the mass generation mechanism.

In this chapter we face the chief point of the mass generation mechanism aiming at outlining a conceptually and mathematically neat framework within which the fundamentals of the Standard Model can be reproduced. This framework is essentially based on the inclusion of the gauge-group parameters into the theory as scalar dynamical fields paralleling the standard Goldstone bosons. With a proper Lagrangian for these new fields of the σ -model type and appropriate rewriting of the traditional Minimal Coupling Prescription we arrive at a general Massive Gauge Theory explicitly exhibiting gauge symmetry. When applied to the electroweak symmetry the new prescription provides mass to the $W^{(\pm)}$ and Z vector bosons without the need for the Higgs particle, leaving naturally the electromagnetic field massless. It might also be used to address low energy effective models for the strong interaction according to the schemes handled in Refs. [49].

The explicit use of the Goldstone bosons in the description of physical processes is by no means new in the literature. In fact, as a consequence of the widely named "Equivalence Theorem" [97, 98], according to which a very heavy Higgs particle can be eliminated from the broken symmetry programme in favour of non-linear σ -like Goldstone bosons, the actual computation of Feynman diagrams involving the longitudinal polarizations of the (massive) vector bosons in electroweak interactions can be resolved in terms of the corresponding diagrams among those scalar fields. But even more, the possibility of incorporating explicitly the Goldstone bosons into the theory as some sort of matter fields has also been considered in the framework of non-abelian (generalized) Stueckelberg theory without Higgs [53]. Unfortunately, the use of a non-linear σ -Lagrangian, as a trace over the whole gauge group, has led to an insoluble dichotomy unitarity-renormalizability [48, 99, 56, 59, 62] (see also the review [46] and references therein).

We will introduce a simple, though essential, modification to the non-abelian Stueckelberg model. We shall adopt a non-linear partial-trace σ -model Lagrangian on G/H instead of on the whole G . The minimal coupling of the new (Goldstone-like) scalar bosons

provides mass terms to those intermediate vector bosons associated with the quotient G/H , without spoiling gauge invariance, so that the H -vector potentials remain massless in a natural way. The advantage of considering a partial trace on G/H , rather than on the entire G , lies on the existence of an infinite-dimensional symmetry enlarging the gauge symmetry group, providing as many non-zero Noether invariants as field degrees of freedom in the solution manifold of the physical system. This ensures quantum integrability, at least under a non-canonical quantization scheme based on symmetry grounds, as has been widely demonstrated in those systems bearing enough symmetries as happens in, for instance, conformal field theories.

It is well known that the non-linear sigma model, in general, suffers from unavoidable renormalizability problems under the canonical quantization programme (see, for instance, [100]). In fact, the trouble that canonical quantization faces in dealing with systems bearing non-trivial topology could be traced back to the “tangent space” approximation imposed at the very beginning of the (canonical) quantization program [103]. Already in the simple case of “free” particles moving on spheres, a proper quantization requires the replacement of canonical commutators with the Lie-algebra commutators of the Euclidean group (see previous chapter; see also [103, 43]). Going further in this direction, we shall replace canonical commutators between coordinates and momenta with Lie-algebra commutators between group generators of the enlarged local symmetry. In fact, the new “canonical” structure of the solution manifold can be derived directly from the symmetry group as one of its canonical invariant forms (giving the symplectic potential).

Let us sketch the usual approach including the Higgs-Kibble mechanism in next section.

4.1 Standard Model of Weak interaction and Higgs mechanism

4.1.1 Gauge invariance and minimal coupling

The Principle of Gauge invariance establishes that interactions are recovered imposing that the internal (rigid) symmetries of a given free theory turn into local ones. According to this, let us consider a matter Lagrangian $\mathcal{L}_{\text{matt}} = \mathcal{L}_{\text{matt}}(\psi, \partial_\mu \psi)$, which is invariant under the action of a given symmetry group G , acting unitarily on the internal states of its fields ψ through the infinitesimal action $\delta_{(a)} \psi^\alpha \equiv T_{(a)\beta}^\alpha \psi^\beta$ ($a = 1, \dots, \dim G$). This group will characterize the obtained interaction. We then impose that this original, rigid symmetry becomes a *local* or *gauge* one, generated by the local group $G(\vec{x}, t)$, the parameters of which now depend on space-time coordinates¹. It is therefore necessary the introduction of new fields, the gauge potentials A_μ . The specific coupling with matter fields is achieved by modifying all derivatives of matter fields in the Lagrangian, $\partial_\mu \psi$,

¹Conceptually, this demand corresponds to the idea that two observers must see the same physics, regardless their gauge choice. Although this rationale is superfluous if we adopt the requirement of gauge invariance as a principle.

with an additive term of the form $qA_\mu^{(a)}T_{(a)}\psi$. This derivative $D_\mu\psi \equiv \partial_\mu\psi - qA_\mu^{(a)}T_{(a)}\psi$ is the so-called covariant derivative, associated with the connection $\Gamma_\mu \equiv A_\mu^{(a)}T_{(a)}$. This is the minimal coupling. The word “minimum” refers to the fact that the gauge field is coupled to the matter fields only through its derivative. The non-tensorial behavior of the gauge field (a connection) under local symmetry transformations compensates the bad transformation properties of conventional derivatives, so that the new Lagrangian is gauge invariant. The Lagrangian governing the dynamics of the gauge fields themselves, \mathcal{L}_0 , must be such that it depends on A_μ and derivatives through the tensorial object $F_{\mu\nu}^{(a)} \equiv \frac{1}{q}[D_\mu, D_\nu]^{(a)} = A_{\mu,\nu}^{(a)} - A_{\nu,\mu}^{(a)} + \frac{q}{2}C_{bc}^a(A_\mu^{(b)}A_\nu^{(c)} - A_\nu^{(b)}A_\mu^{(c)})$ (where C_{bc}^a are the structure constants of the Lie algebra of the rigid symmetry group), also known as the curvature of the connection associated with gauge potentials.

This prescription provides a gauge theory where bosons mediating the interaction are massless. If mass terms are put by hand, gauge invariance is broken. In the Abelian case, the symmetry group being $U(1)$, the electrodynamics is obtained with a massless photon, in agreement with phenomenology. However, in the electroweak interaction, with gauge group $SU(2) \otimes U(1)$, the particles W^\pm y Z^0 possess mass.

In order to find agreement with such circumstance keeping the power of a gauge theory, a mass-generating mechanism is required to build a gauge invariant gauge theory.

4.1.2 Spontaneous symmetry breaking

The most widely accepted solution to mass generation, in the context of the Standard Model of Particle Physics, makes use of the spontaneous symmetry breaking phenomenon, based on the fact that the solutions of basic equations describing a physical system need not to have the same set of symmetries as the equations themselves. This is applied, in particular, to the ground state of the system.

Let us consider a Lagrangian describing a multiplet of self-interacting, real scalar fields $\Phi_i(x)$, with potential $V(\Phi)$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu\Phi_i) (\partial^\mu\Phi_i) - V(\Phi_i),$$

invariant under the action of a global group G on $\Phi_i(x)$:

$$\delta^a\Phi_i(x) = T_{ij}^a\Phi^j(x),$$

where $T_{ij}^a, a = 1, \dots, N$ are the N generators of G . Then,

$$0 = \delta^a V = \frac{\partial V}{\partial\Phi_i} \delta^a\Phi_i = \frac{\partial V}{\partial\Phi_i} T_{ij}^a\Phi^j = 0.$$

Differentiating we find that the invariance of \mathcal{L} under G requires

$$\frac{\partial^2 V}{\partial\Phi_i\partial\Phi_k} T_{ij}^a\Phi^j + \frac{\partial V}{\partial\Phi_i} T_{ik}^a = 0.$$

Let us consider the case in which the potential V has a minimum in $\Phi_i \equiv v_i \neq 0$, that is to say, the ground state (corresponding to the vacuum expectation value of the quantum

system) does not have zero energy. Then

$$\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} \Big|_{\Phi_i = v_i} T_{ij}^a v^j = 0, \quad (4.1)$$

and for small oscillations around the vacuum state, we can write

$$V(\Phi) = V(v_i) + \frac{1}{2} \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} \Big|_{\Phi_i = v_i} (\Phi_i - v_i)(\Phi_k - v_k) + \dots$$

It is then obvious that

$$(M^2)_{ik} = \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} \Big|_{\Phi_i = v_i}$$

are mass terms for the shifted fields $\Phi'_i = \Phi_i - v_i$.

In general, a subgroup H of G can exist under which the ground state is invariant

$$\hat{T}_{ij}^a v^j = 0, \quad \hat{a} = 1, \dots, \hat{N}$$

Then, \hat{N} of the N equations in (4.1) are trivially satisfied and $M^2 > 0$. The rest of the $N - \hat{N}$ equations in (4.1) require eigenvalues $M^2 = 0$. The corresponding field excitations are called *Goldstone bosons*.

4.1.3 Higgs-Kibble mechanism in the Standard Model

Let us see what happens if we consider a theory with a non-Abelian local gauge symmetry, where scalar fields Φ_i are incorporated, minimally coupled to the gauge vector bosons.

The gauge bosons corresponding to the broken symmetry become massive and Goldstone bosons do not appear. This means that the Goldstone modes are transformed into the longitudinal components of the gauge fields.

Let us consider a self-interacting complex scalar field (the Higgs field), represented by a doublet of $SU(2)_L$ (here L indicates that this symmetry acts only on fermions with left-handed chirality):

$$\Phi(x) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

which is invariant under local transformations of the symmetry group $SU(2)_L \otimes U(1)_Y$

$$\mathcal{L}_H = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi)$$

$$V(\Phi) = -\mu^2 (\Phi^\dagger \Phi) + \lambda (\Phi^\dagger \Phi)^2, \quad \lambda > 0$$

$$D_\mu = \partial_\mu + i g \frac{\tau^a}{2} W_\mu^a + i g' \frac{Y_H}{2} B_\mu.$$

The Spontaneous symmetry breaking (SSB) takes place when we choose the sign $\mu^2 > 0$ for the mass term. The desired SSB pattern is:

$$SU(2)_L \otimes U(1)_Y \longrightarrow U(1)_{em},$$

in such a way that three gauge vector bosons become massive and the one corresponding to $U(1)_{em}$, the photon, remains massless (the vacuum state is invariant under $U(1)_{em}$). Thus, we choose fields and corresponding charges $\Phi_2 = \Phi^0$ with $Q = 0$ and $Y_H = \frac{1}{2}$, and also $\Phi_1 = \Phi^+$ with $Q = 1$ and $Y_H = \frac{1}{2}$.

The potential V minimizes for a constant field $\Phi^\dagger \Phi = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$. To a first approximation, the expectation value of the quantum system is given by the minimum energy state of the classical potential

$$|\langle 0|\Phi|0\rangle| = \frac{v}{\sqrt{2}}; \quad v = \frac{\mu}{\sqrt{\lambda}}$$

The orientation of the ground state in the weak isospin space is not fixed

$$|\langle 0|\Phi|0\rangle| = e^{iT^a \xi_a} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

It is the choice of a ground state what spontaneously breaks the $SU(2)$ symmetry.

Let us consider vacuum excitations and lets perform the substitution $\Phi' = \Phi - v$. By construction $|\langle 0|\Phi'|0\rangle| = 0$. Then we see that the combination $\Phi_2'^\dagger + \Phi_2'$ acquires mass

$$V(\Phi') \approx \lambda \frac{v^2}{2} (\Phi_2'^\dagger + \Phi_2')^2,$$

and the other three excited modes of the Higgs field remain massless (Goldstone bosons).

Let us see what happens with these Goldstone modes when the Higgs field minimally couples to the gauge bosons. As the Higgs field is described by a $SU(2)$ -invariant Lagrangian, we can parametrize the Higgs field as follows:

$$\Phi(x) = e^{iT^a \xi_a(x)} \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}$$

with $|\langle 0|\xi_a(x)|0\rangle| = 0$ and $|\langle 0|\eta(x)|0\rangle| = 0$. Then, choosing the *unitary gauge*, where the would-be Goldstone bosons ξ_a do not appear:

$$\Phi(x) = \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}.$$

Then, performing a change to the ‘‘spherical’’ basis in the gauge fields and the Weinberg rotation to arrive to the physical fields (mass eigenstates), we find mass terms for the vector bosons W^\pm and Z^0 :

$$\mathcal{L}_H = M_W^2 W^{\mu-} W_\mu^+ + \frac{1}{2} M_Z^2 Z_\mu Z^\mu - \frac{1}{2} M_\eta^2 \eta^2 + \dots$$

$$M_W = \frac{vg}{2} \quad M_Z = \frac{v}{2} \sqrt{g^2 + g'^2},$$

and no mass term for the photon A_μ . The choice $g \sin \theta_w = g' \cos \theta_w$ has been made. The Goldstone bosons do not appear: it is said that the gauge bosons ‘‘eat’’ that degree of freedom to give rise to their longitudinal mode. A physical scalar field survives, the Higgs boson, with $M_\eta = \frac{\sqrt{\lambda}}{2} v$. In addition, mass terms for the fermions can be obtained in an invariant way through Yukawa couplings to the Higgs field.

4.2 Stueckelberg mechanism

The Stueckelberg [51] provides a way of implementing gauge invariance in electrodynamics with a massive photon, with no need of a spontaneous symmetry breaking mechanism. The generalization of the Stueckelberg model to the non-Abelian case would constitute, in principle, an alternative to the Higgs-Kibble mechanism in the Standard Model. The quantum Abelian version is unitary and renormalizable in the conventional sense. This was proven rigorously in [52]. However, we will see that this seems to be no longer true for the non-Abelian generalization usually considered in the literature [53].

The idea in the original Stueckelberg model was to introduce a scalar field B , that compensates the bad transformation properties of the gauge fields mass terms. The Lagrangian for massive electrodynamics, as described by the Stueckelberg model (with no matter fields), is

$$\mathcal{L} = \frac{1}{4}F^2 + \frac{M^2}{2} (A_\mu - \partial_\mu B)^2,$$

which is obtained from the Proca Lagrangian by means of the shift (mimicking a gauge transformation) $A_\mu \rightarrow A_\mu - \partial_\mu B$. This does not affect the form of the coupling with matter fields because, due to gauge invariance, such shift can be compensated. This Lagrangian is now gauge invariant under transformations

$$\delta A_\mu = \partial_\mu \varphi,$$

provided that the new field transforms as the gauge parameter itself, that is

$$\delta B = \delta \varphi.$$

We see that two elements in the theory behave the same way: the Stueckelberg field $B(x)$ and the $U(1)$ gauge group parameter, $\varphi(x)$.

The natural non-Abelian extension of the Stueckelberg formalism for a general gauge group G follows similar steps to the Abelian case [53]. Now $U(x) = e^{i\varphi^a(x)T_a} \in G(M)$, where $T_a, a = 1, \dots, \dim(G)$ are the Lie-algebra generators of G with commutation relations $[T_a, T_b] = C_{ab}^c T_c$. We shall restrict ourselves to unitary groups and set the normalization $\text{Tr}(T_a T_b) = \delta_{ab}$. When referring to the canonical 1-form on G , we must distinguish between the left- and right-invariant ones: $\theta_\mu^L = -iU^\dagger \partial_\mu U$ and $\theta_\mu \equiv \theta_\mu^R = -i\partial_\mu U U^\dagger$, respectively. The G -invariant σ -model Lagrangian now reads:

$$\mathcal{L}_\sigma^G = \frac{1}{2}\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) = \frac{1}{2}\text{Tr}(\theta_\mu \theta^\mu) = \frac{1}{2}\text{Tr}(\theta_\mu^L \theta^{L\mu}) \equiv \frac{1}{2}g_{ab}(\varphi)\partial_\mu \varphi^a \partial^\mu \varphi^b \quad (4.2)$$

which is highly non-linear and chiral. The minimal coupling is formally analogous to the Abelian case, namely

$$\mathcal{L}_\sigma^G = \frac{1}{2}\text{Tr}((D_\mu U)(D^\mu U)^\dagger) = \frac{1}{2}\text{Tr}((\theta_\mu - A_\mu)(\theta^\mu - A^\mu)), \quad (4.3)$$

although A_μ must be understood as $A_\mu = A_\mu^a T_a$. This Lagrangian is invariant, in particular, under

$$U \rightarrow VU, A_\mu \rightarrow VA_\mu V^\dagger - i\partial_\mu V V^\dagger. \quad (4.4)$$

Adding the standard kinematical Lagrangian for Yang-Mills fields $\mathcal{L}_{\text{YM}}^G = -\frac{1}{4}\text{Tr}(F^{\mu\nu}F_{\mu\nu})$, with

$$F_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (4.5)$$

to (4.3), we arrive at the full Lagrangian for Massive Yang-Mills bosons

$$\mathcal{L}_{\text{MYM}}^G = \mathcal{L}_{\text{YM}}^G + m^2 \mathcal{L}_\sigma^G. \quad (4.6)$$

As already mentioned in the introduction, this model prevents the massive Yang-Mills theory from being both unitary and renormalizable, at least in the canonical quantization approach.

4.2.1 Unitarity and renormalizability debate

The Stueckelberg model, in its non-Abelian version, has been object of a big amount discussions in the literature², mainly to determine whether it constitutes a good mass-generating mechanism for the electroweak intermediate vector bosons. Many authors feel that unitarity and renormalizability can not be established simultaneously. Roughly speaking, the reason is the non-polynomial form in terms of the Stueckelberg fields of the Lagrangian, so that renormalizability can not be determined in conventional terms. Transformations leading to a polynomial form break unitarity (i.e., they lead to a non-equivalent system).

Given that massless Yang-Mills theories are renormalizable, one could expect that so are the corresponding massive theories if the limit $m \rightarrow 0$ exists. However, this limit is singular. The reason for this singularity can be understood easily by the counting of degrees of freedom: a massive vector particle has three field degrees of freedom, while a massless one has only two.

In [48] an overview of different versions of the Stueckelberg model is given, focusing on the conflicts between unitarity and renormalizability. The first example is the one given in [53], supported by [65] y [66]. In this last reference, they establish the unitarity of the gauge boson propagator to one loop in perturbation theory, working on the Landau gauge. It is explicitly shown that the compensation for the imaginary part of this propagator comes from the Faddeev-Popov ghosts contribution as well as the Stueckelberg field contribution. This means that, in the limit of zero mass, the massless Yang-Mills theory is not recovered, as already commented. This analysis is repeated for the fermion-antifermion scattering till order g^4 by Delbourgo et al. in [48], finding again that the Stueckelberg scalars contributes with the correct sign.

Then, they study the high-energy behavior of longitudinally polarized vector bosons, computing their inelastic scattering. In a theory with Higgs boson the corresponding amplitude is bounded, in agreement with unitarity. For the Stueckelberg case, even if $S^\dagger S = 1$ is satisfied order by order in g^2 , it turns out that the amplitude in increasing orders scales with increasing powers of $\frac{E^2}{m^2}$. They conclude that renormalizability is not perturbatively satisfied in the generalized Stueckelberg scheme. However, [48] points out that

²In this subsection we will follow the review by Ruegg [46] and references therein, and also [48].

in [54] renormalizability of two-dimensional massive Yang-Mills theory was established in a non-conventional way.

A more complete discussion about limits on unitarity can be found in [55]. There, the concept of tree-unitarity is introduced. They highlight that “an important advantage of the Stueckelberg formalism is that the whole bad behavior can be isolated in the vertexes. They conclude that tree-unitarity is only satisfied in models with spontaneous symmetry breaking.

[48] concludes that renormalizability and unitarity seem to be confronted characteristics of non-Abelian massive theories: “The original Stueckelberg formulation, with its inherent non-polynomiality, is unitary but not renormalizable. This is in itself quite interesting, implying that the naive massive Yang–Mills action is of the correct form to ensure unitarity, and as we have seen any tampering with this leads us astray”.

The non-Abelian Stueckelberg Lagrangians contain non-polynomial terms in the canonically quantized fields. Maybe a correct understanding of these terms could lead to a unitary and renormalizable Stueckelberg model. The work by Dragon et al. [58] discards this possibility showing that the non-polynomial structure can be algebraically reduced to a polynomial version of the Stueckelberg model. But then results by Hurth in 1997 [59] can be applied: it turns out that this model is non-unitary but renormalizable.

This last attempt illustrates the efforts to modify the theory so that it is tractable with conventional methods based on Canonical Quantization and subsequent perturbation theory. Maybe this is not the correct way. As said in [61]: “It is quite plausible that the physical sector of massive Yang–Mills theory be renormalizable although it is not by power counting and although it is not unitary at each order of the perturbation expansion. This is particular to massive Yang–Mills theory and the existence of such a renormalizability is still an open question which merits further attention especially if Higgs bosons remain experimentally undetected.”

As a concluding observation, we mention the open possibility in the work by Delbourgo et al. [48]: “(...) we see that in such desired gauges as the Landau gauge, the boson propagator is well behaved, however the action remains nonpolynomial. The theory turns out not to be conventionally renormalizable, though there is some hope that if one moves outside of conventional perturbation theory, by using the methods introduced by Efimov [60] for dealing with nonpolynomial interactions, sense may still be made of the theory.”

4.3 Quantum field theory with symmetries

Toward a group-theoretical formulation of the Stueckelberg model, we now outline the general scheme to deal with field theories in GAQ. Typical infinite-dimensional systems in Physics appear as mappings from a space-time manifold M into a (not necessarily Abelian) target group G

$$\varphi : M \rightarrow G, \quad x \mapsto \varphi(x). \quad (4.7)$$

If a is an invertible, differentiable transformation of M , i.e. a is an element in $\text{Diff}(M)$,

or a subgroup of it, the following semi-direct ($\text{Diff}(M) \otimes_s G(M)$) group law holds:

$$a'' = a' \circ a, \quad \varphi''(x) = \varphi'(a(x)) * \varphi(x), \quad (4.8)$$

where \circ is the composition group law in $\text{Diff}(M)$ (composition of mappings), $*$ denotes the composition group law in the target group G and $a(x)$ stands for the action of $\text{Diff}(M)$ on M . When the group G is not a (complex) vector space \mathbb{C}^n , the group of mappings is usually called gauge, local or current group $G(M)$. Specially well-known are the unitary gauge groups on Minkowski space-time and the loop groups which correspond to the case in which M is the circle S^1 . However, the actual physical fields correspond to the elements in the centrally extended group $\widetilde{\text{Diff}(M)} \otimes_s G(M)$. In the case of $M = S^1$ the group (4.8) has a specially rich structure (that is Virasoro \otimes_s Kac-Moody) [36] with many applications in conformal field theory in 1 + 1 dimensions [37]. As a general comment, the ability in parameterizing the infinite-dimensional group (4.8) will play a preponderant role in the corresponding physical description.

4.3.1 The Klein-Gordon Field

As a very simple example of the general scheme above-mentioned let us consider the case in which M is the Minkowski space-time parameterized by $(x^0 \equiv ct, \vec{x})$, $\text{Diff}(M)$ is restricted to its Poincaré subgroup (or even just the space-time translations subgroup, for the sake of simplicity), parameterized by $(a^0 \equiv cb, \vec{a})$, and G is simply a complex vector space, let us say \mathbb{C} , parameterized by φ .

There is a natural parameterization of the group above associated with a factorization of M as $\Sigma \times R$, that is, a Cauchy surface Σ times Time. In fact, we can use $\langle b, \vec{a}; \varphi(\vec{x}), \dot{\varphi}(\vec{x}) \rangle$. In these variables, however, whereas the space translations \vec{a} act on $\varphi(\vec{x})$ by just moving the arguments as $\vec{x} + \vec{a}$: $\varphi(\vec{x} + \vec{a}) = \exp(i\vec{a} \cdot \partial_{\vec{x}})\varphi(\vec{x})$, making b an action on $\varphi(\vec{x}), \dot{\varphi}(\vec{x})$ requires the knowledge of the equation of motion, though not necessarily their solutions. For the Klein-Gordon field the time evolution equations are

$$\ddot{\varphi}(\vec{x}) = (\vec{\nabla}^2 - m^2)\varphi(\vec{x})$$

and the time action $\varphi'(b(\vec{x}))$ reads:

$$\varphi'(b(\vec{x})) \equiv e^{ibc\hat{a}_0}\varphi'(\vec{x}) = \cos \left[bc\sqrt{m^2 - \vec{\nabla}^2} \right] \varphi'(\vec{x}) + i \frac{\sin \left[bc\sqrt{m^2 - \vec{\nabla}^2} \right]}{\sqrt{m^2 - \vec{\nabla}^2}} \dot{\varphi}'(\vec{x})$$

This way, all canonical operations on groups can be easily performed. For instance, the

right-invariant vector fields are:

$$X_b^R = \frac{\partial}{\partial b}$$

$$X_{\varphi(\vec{x})}^R = \cos\left(b\sqrt{m^2 - \vec{\nabla}^2}\right) \frac{\partial}{\partial \varphi(\vec{x})} - \sqrt{m^2 - \vec{\nabla}^2} \sin\left(b\sqrt{m^2 - \vec{\nabla}^2}\right) \frac{\partial}{\partial \dot{\varphi}(\vec{x})}$$

$$X_{\dot{\varphi}(\vec{x})}^R = \cos\left(b\sqrt{m^2 - \vec{\nabla}^2}\right) \frac{\partial}{\partial \dot{\varphi}(\vec{x})} + \frac{1}{\sqrt{m^2 - \vec{\nabla}^2}} \sin\left(b\sqrt{m^2 - \vec{\nabla}^2}\right) \frac{\partial}{\partial \varphi(\vec{x})},$$

and their commutation relations:

$$\left[X_b^R, X_{\varphi(\vec{x})}^R\right] = -(m^2 - \vec{\nabla}^2)X_{\varphi(\vec{x})}^R$$

$$\left[X_b^R, X_{\dot{\varphi}(\vec{x})}^R\right] = X_{\varphi(\vec{x})}^R$$

$$\left[X_{\varphi(\vec{x})}^R, X_{\dot{\varphi}(\vec{x}')}\right] = 0 \quad (\delta(\vec{x} - \vec{x}')X_{\dot{\varphi}}^R \text{ when centrally extended})$$

Notice that the actual solutions of the equations of motion of a more general system are not required since the corresponding Lie algebra can be exponentiated (at least) order by order giving rise to the finite action of b on both $\varphi(\vec{x})$ and $\dot{\varphi}(\vec{x})$.

As mentioned above, what really matters for the physical description is the corresponding centrally extended group. In order to motivate such extension we shall proceed in a way analogous to that followed in the case of Mechanics. Let us go then temporarily to the standard Lagrangian formalism for classical fields. The real Klein-Gordon field of mass m is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) \quad (4.9)$$

which is well-known to realize the Poincaré symmetries (see, for instance Ref. [38]). However, the Noether invariants associated with space-time symmetries are not relevant in studying the solution manifold \mathcal{M} in the sense that they are not the basic, independent functions parameterizing the phase space. In fact, quantities such as the energy-momentum tensor or the generalized (rotations and Lorentz) angular momenta are written in terms of the Fourier coefficients $a(k)$, $a^*(k)$, where the four vector k_μ runs on the Lorentz orbit $k^\mu k_\mu = m^2$. Here we are primarily interested in characterizing those Fourier coefficients as Noether invariants of certain generators leaving semi-invariant the Lagrangian (4.9). To this end we consider the following vector fields on the complete (including the field derivatives) configuration space for the Klein-Gordon Field $(x^\nu, \varphi, \varphi_\mu)$:

$$\bar{X}_{a^*(k)} \equiv i e^{ikx} \frac{\partial}{\partial \varphi} - k_\nu i e^{ikx} \frac{\partial}{\partial \varphi_\nu} \quad (4.10)$$

Computing the Lie derivative of the Lagrangian with respect to this vector (note that the second components of this vector are simply the derivatives of the components on φ) we obtain:

$$L_{\bar{X}_{a^*(k)}} \mathcal{L} = \partial_\mu \beta^\mu, \quad \beta^\mu \equiv -k^\mu e^{ikx} \varphi, \quad (4.11)$$

where explicit use of the mass-shell condition for k has been made. The Noether theorem establishes that the current

$$J_{a^*(k)}^\mu = X_{a^*(k)}^\varphi \pi^\mu - \beta^\mu$$

(where $X_{a^*(k)}^\varphi$ is the φ -component of the generator, i.e. the infinitesimal variation $\delta\varphi$ and $\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \varphi_\mu}$ is the field covariant momentum) is conserved: $\partial_\mu J^\mu = 0$. The Noether charge reads:

$$Q_{X_{a^*(k)}} = \int d^3x J^0 = i \int d^3x e^{ikx} (\dot{\varphi} - ik^0 \varphi) \quad (4.12)$$

which turns out to be just the Fourier coefficient $a(k)$. In the same way we obtain the charge $a^*(k)$ and so a coordinate system for the solution manifold \mathcal{M} made of Noether invariants.

Analogously, an equivalent configuration-space parameterization can be considered. In fact, the vector fields:

$$X_{\pi(\vec{y})} \equiv i \int \frac{d^3k}{2k^0} \left[e^{i\vec{k}\cdot\vec{y}} e^{ikx} \frac{\partial}{\partial \varphi} - h.c. \right] \quad (4.13)$$

$$X_{\varphi(\vec{y})} \equiv - \int \frac{d^3k}{2k^0} k^0 \left[e^{i\vec{k}\cdot\vec{y}} e^{ikx} \frac{\partial}{\partial \varphi} + h.c. \right]$$

have as Noether invariants the values of φ and $\pi = \dot{\varphi}$ on each one of the points of the Cauchy surface Σ . In terms of these "configuration-space" variables the symplectic form on \mathcal{M} adopts the aspect and properties of that of Classical Mechanics:

$$w_{K-G} = d\Lambda_{K-G} = d \left(\int d^3x \pi(\vec{x}) d\varphi(\vec{x}) \right),$$

the symplectic potential Λ_{K-G} , or Liouville form, being *semi-invariant under the basic symmetries* (4.13).

To end up with the (semi-)invariance properties of the Klein-Gordon field it should be mentioned that the symmetries of this Lagrangian can be given the aspect of some sort of "residual gauge" symmetry, even for $m \neq 0$. In fact, for any real function f on M satisfying the Klein-Gordon equation, the vector field

$$X^f = f \frac{\partial}{\partial \varphi} + f_\mu \frac{\partial}{\partial \varphi_\mu}$$

leaves (4.9) semi-invariant.

Once the necessity of a central extension of the semi-invariance group (4.8) has been stated and motivated, we write the quantization group for the Klein-Gordon field in covariant form [39] as follows:

$$a'' = a' + \Lambda' a$$

$$\Lambda'' = \Lambda' \Lambda$$

$$\varphi''(x) = \varphi'(\Lambda x + a) + \varphi(x)$$

$$\varphi''_{\mu}(x) = \varphi'_{\mu}(\Lambda x + a) + \varphi_{\mu}(x)$$

$$\zeta'' = \zeta' \zeta \exp \left\{ \frac{i}{2} \int_{\Sigma} d\sigma^{\mu} \left[\varphi'(\Lambda x + a) \varphi_{\mu}(x) - \varphi'_{\mu}(\Lambda x + a) \varphi(x) \right] \right\}$$

The exponential is well-defined, even though the fields have been written on the whole space-time, because the integrand is a conserved current. Further details can be found in [39] and references therein.

4.4 Group-theoretical treatment of the Stueckelberg model

Now we turn to the Stueckelberg mechanism with the eyes of a group-theorist. From the point of view of GAQ, the idea is to give dynamics to the gauge group parameters. That is the way we introduce the Stueckelberg fields: they are the living, dynamical part of the gauge group parameters.

A Group quantization of non-Abelian gauge groups had been only achieved consistently in $1 + 1$ dimensions by representing the corresponding Kac-Moody group [36]. In fact, the special structure (non-trivial cohomology) of such groups allow for a central extension providing a quantum representation of the Poisson algebra associated with a WZW-type Lagrangian. In $3 + 1$ dimensions, however, the Mickelsson central extension (two-cocycle) is absent and a bit more involved construction is required. One of the new required ingredients will be the consideration of an enlarged symmetry group $G^1(M)$, containing the gauge group $G(M)$ (the standard gauge group only contribute with null Noether invariants) parametrized by the Goldston-like scalar fields $\varphi^a(x)$, along with the corresponding vector potentials $A_{\mu}^a(x)$ parametrizing the rest of the new group [64]. The other ingredient refers to the use of a class of central extensions (two-cocycles) that, even though they are trivial from some mathematical points of view, they define central extensions of the group (and select therefore specific projective representations of the unextended group) endowed with a canonical (left- or right-) invariant form which gives a physical Lagrangian for fields living on a coadjoint orbit of $G^1(M)$. The corresponding Lagrangian can then be seen as a (covariant) partial-trace of the standard σ -model full-trace (chiral) Lagrangian $Tr(U^{-1} \partial_{\mu} U U^{-1} \partial^{\mu} U)$, $U \in G$, coupled to the vector potentials

according to a Minimal Coupling prescription addressed by the proper structure of the central extension of the local group. The fact that the Minimal Coupling in these partial-trace Lagrangians/groups, leaves the vector potentials associated with the subgroup H addressing the co-adjoint orbit G/H massless, makes the mechanism specially suited to describe alternatives to the Standard Model without Higgs particles.

4.4.1 Partial-trace massive non-Abelian Yang-Mills system: classical theory

4.4.1.1 Lagrangian formalism

With respect to the Stueckelberg model described above, based on a total trace sigma model Lagrangian, the situation is soundly improved by restricting the whole trace on G to a partial trace on a quotient manifold G/H . H is the isotropy subgroup of a given direction $\lambda = \lambda^a T_a$, in the Lie-algebra of G , under the adjoint action $\lambda \rightarrow V\lambda V^\dagger$, where λ^a are real numbers subjected to $\text{Tr}(\lambda^2) = 1$. From a strict group-theoretical point of view and thinking of our specific quantization technique, the main advantage of dealing with partial-trace Lagrangians is that they, or the corresponding Poincaré-Cartan (also named Hilbert or canonical) forms [40, 41, 42], can be derived from a centrally-extended Lie group in much the same way the Lagrangian, and the entire (quantum) theory of a free particle can be derived from a $U(1)$ central extension of the Galilei group. And this fact might be related to the particular fact that partial-trace Lagrangians can be written as the square of a total derivative. In fact, by defining $\Lambda \equiv U\lambda U^\dagger$, the claimed $G/H - \sigma$ Lagrangian can be written with the following expression:

$$\mathcal{L}_\sigma^{G/H} = \frac{1}{2}\text{Tr}([-iU^\dagger\partial_\mu U, \lambda]^2) \equiv \frac{1}{2}\text{Tr}([\theta_\mu^L, \lambda]^2) = \frac{1}{2}\text{Tr}([\theta_\mu, \Lambda]^2) = \frac{1}{2}\text{Tr}((\partial_\mu \Lambda)^2). \quad (4.14)$$

Let us proceed with its minimally coupled version:

$$\tilde{\mathcal{L}}_\sigma^{G/H} = \frac{1}{2}\text{Tr}([-iU^\dagger D_\mu U, \lambda]^2) = \frac{1}{2}\text{Tr}([\theta_\mu - A_\mu, \Lambda]^2), \quad (4.15)$$

which is again gauge invariant under (4.4). As in (4.6), the partial-trace (G/H) Massive Yang-Mills Lagrangian now follows:

$$\mathcal{L}_{\text{MYM}}^{G/H} = \mathcal{L}_{\text{YM}}^G + m^2 \tilde{\mathcal{L}}_\sigma^{G/H}. \quad (4.16)$$

We should remark that the change of variables

$$\tilde{A}_\mu = U^\dagger(A_\mu - \theta_\mu)U = U^\dagger A_\mu U + iU^\dagger\partial_\mu U, \quad (4.17)$$

and the fact that $F(A) = UF(\tilde{A})U^\dagger$, renders the Lagrangian (4.16) into the simple form

$$\mathcal{L}_{\text{MYM}}^{G/H} = -\frac{1}{4}\text{Tr}(F^{\mu\nu}(\tilde{A})^2) + \frac{1}{2}m^2\text{Tr}([\tilde{A}_\mu, \lambda]^2). \quad (4.18)$$

This change of variables, formally mimicking the shift to the unitary gauge, turns the actual degrees of freedom of the theory apparent; that is to say, those of $\dim H$ massless vector fields (in the λ direction) and $(n - \dim H)$ massive ones. On the other hand, it must be eventually completed with the change of variables $\phi = U^\dagger\psi$ when the fermionic matter field ψ will be introduced.

4.4.1.2 Symplectic structure

Linear systems provide a Poisson algebra of classical observables in the solution manifold realizing the Lie algebra of the Heisenberg-Weyl group in the corresponding dimension. In field theories, this suggests to approach the corresponding quantum theory by postulating equal-time commutation relations between fields and their time derivatives, or conjugate momenta, $[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y})$. Going to non-linear systems with non-flat phase space should require a different approach [103, 100]. This is precisely the situation we are facing now, as a result of the introduction of the group parameters as physical degrees of freedom, with a (curved) compact target space G/H . We must look for a replacement of the Heisenberg-Weyl group with a (more involved) symmetry group of the solution manifold, keeping the general idea of considering as basic conjugate coordinates those giving central terms under commutation. Therefore, we should be able to identify such symmetry group by analyzing the symplectic potential (or Liouville 1-form) in the solution manifold, which generalizes $J \equiv p_i dq^i$ from particle mechanics. The symplectic potential for a generic scalar field ϕ can be obtained by integrating the Lagrangian Poincaré-Cartan form

$$\Theta_{PC} \equiv \frac{\partial \mathcal{L}}{\partial \phi_\mu} (d\phi - \phi_\nu dx^\nu) \wedge d\sigma_\mu + \mathcal{L} d^4x \quad (4.19)$$

on a Cauchy hypersurface Σ , leading to the canonical, Darboux-like, expression

$$J^\mathcal{L} = \int_\Sigma J_\mu^\mathcal{L} d\sigma^\mu = \int_\Sigma \pi_\mu \delta\phi d\sigma^\mu, \quad (4.20)$$

that becomes $\int_\Sigma \pi(\vec{x}) \delta\phi(\vec{x}) d^3x$ for the hypersurface $x^0 = \text{const}$, where $\pi(\vec{x}) \equiv \pi_0(\vec{x})$ and $\phi(\vec{x})$ are the Noether-invariant field momentum and field coordinate, respectively. The symplectic structure then becomes $\omega \equiv \delta J^\mathcal{L}$.

The expression (4.20) for the symplectic potential is in general of local character and only becomes global for linear systems. In fact, for a Klein-Gordon field ϕ this expression is global and can be derived as the canonical symplectic potential on the co-adjoint orbits of the infinite-dimensional Heisenberg-Weyl-like group law (see the next section for details):

$$\begin{aligned} \phi''(x) &= \phi'(x) + \phi(x) \\ \phi_\mu''(x) &= \phi_\mu'(x) + \phi_\mu(x) \\ \zeta'' &= \zeta' \zeta \exp \left\{ \frac{i}{2} \int_\Sigma d\sigma^\mu \left[\phi'(x) \phi_\mu(x) - \phi_\mu'(x) \phi(x) \right] \right\}, \end{aligned} \quad (4.21)$$

which constitutes a central extension by $U(1)$, parametrized by ζ , $|\zeta|^2 = 1$, of the Abelian group parametrized by “coordinates” $\phi(x)$ and (covariant) “momenta” $\pi_\mu(x) = \partial_\mu \phi(x)$. It will be understood that a quantization of the field $\phi(x)$ is achieved by means of a unitary and irreducible representation of this group.

Coming back to our present system and regarding the sigma sector, which is the typically non-linear part, the symplectic potential can be written as:

$$J_\sigma^\mathcal{L} = \int_\Sigma \pi_a^\mu \delta \varphi^a d\sigma_\mu = \int_\Sigma \text{Tr}([\theta_\mu, \Lambda] [-i \delta U U^\dagger, \Lambda]) d\sigma^\mu = \int_\Sigma \text{Tr}([\theta_\mu, \Lambda] \delta \Lambda) d\sigma^\mu, \quad (4.22)$$

where the conserved momentum current is

$$\pi_a^\mu = \text{Tr}([\theta^\mu, \Lambda] [-i \frac{\partial U}{\partial \varphi^a} U^\dagger, \Lambda]), \quad (4.23)$$

and from (4.14), we may rewrite the Liouville form on G/H as

$$J_\sigma^\mathcal{L}|_{G/H} = \int_\Sigma \text{Tr}([\theta_\mu, \Lambda], \Lambda] \delta \Lambda) d\sigma^\mu. \quad (4.24)$$

The natural question now arises of up to whether or not the potential (4.24) can also be derived from a (non-Abelian) group generalizing the Heisenberg-Weyl one. The new group has to be the basic symmetry of the solution manifold of our partial-trace non-linear sigma model, which must appear as a co-adjoint orbit. In fact, the following group law, with parameters $\varphi^a(x)$, or in matrix form, $U(x) = e^{i\varphi^a(x)T_a}$, playing the role of coordinates, θ_μ^a as corresponding momenta, and $\zeta \in U(1)$ parametrizing the central generator, accomplishes this task:

$$\begin{aligned} U'' &= U'U \\ \theta_\mu'' &= U' \theta_\mu U'^\dagger + \theta_\mu' \\ \zeta'' &= \zeta' \zeta \exp\{i \int_\Sigma \text{Tr} \lambda (U'(x) \theta_\mu(x) U'^\dagger(x) - \theta_\mu(x)) d\sigma^\mu\}. \end{aligned} \quad (4.25)$$

Unlike the group law (4.21), the cocycle addressing the group law for the $U(1)$ subgroup is a coboundary with a non-trivial physical content (see [101, 102] and references therein). Once again, a unitary and irreducible representation of this group will constitute a proper (though non-canonical) quantization of the non-linear sigma fields living on G/H .

The Lie algebra of this symmetry group is intended to provide a faithful representation of the Poisson bracket among the corresponding Noether invariants in the solution manifold. They are (see the next section):

$$\begin{aligned} I_{\phi^a(\vec{x})} &= [\theta_\mu(\vec{x}), \Lambda(\vec{x})] n^\mu(\vec{x}) \\ I_{\theta_\mu^a n^\mu(\vec{x})} &= \Lambda^a(\vec{x}) \equiv \text{Tr}(T^a U(\vec{x}) \lambda U^\dagger(\vec{x})), \end{aligned}$$

where the vector $n^\mu(x)$ characterize the volume on Σ , that is, $d\sigma^\mu = n^\mu d\sigma$, and $\text{Tr}(\Lambda_\mu \Lambda) = 0$, $\Lambda_\mu \equiv [\theta_\mu, \Lambda]$. The zero component of $[\theta_\mu, \Lambda]$, to be named simply \mathbb{L} , plays the role of conserved local (internal) “angular momentum” associated with the “displacement” on

the solution manifold $\delta\Lambda$ or $-i\delta UU^\dagger$. The choice of the Noether invariants Λ and L as coordinates on the solution manifold makes apparent the actual number of degrees of freedom.

Now that we have analyzed to some extent the symplectic structure and symmetries of the σ sector of the theory, the construction of symplectic potential of the complete theory proceeds without special difficulty. In fact, the pure Yang-Mills sector behaves as in the case of a Klein-Gordon field and the mixed terms follow from the minimal coupling replacement. We then have for the complete symplectic potential for the Massive Yang-Mills theory the expression:

$$J_{MYM}^{\mathcal{L}} = \int_{\Sigma} \text{Tr}(F^{\mu\nu}(\tilde{A})\delta\tilde{A}_\nu - m^2[A^\mu - \theta^\mu, \Lambda]\delta\Lambda)d\sigma_\mu, \quad (4.26)$$

which can be rewritten, on account that it is living on the orbit G/H , as

$$J_{MYM}^{\mathcal{L}}|_{G/H} = \int_{\Sigma} \text{Tr}(F^{\mu\nu}(\tilde{A})\delta\tilde{A}_\nu - m^2[[A^\mu - \theta^\mu, \Lambda], \Lambda]\delta\Lambda)d\sigma_\mu \quad (4.27)$$

$$\approx \int_{\Sigma} \text{Tr}(F^{\mu\nu}(\tilde{A})\delta\tilde{A}_\nu + m^2\tilde{A}^\mu[U^\dagger\delta U, \lambda])d\sigma_\mu, \quad (4.28)$$

where in the last line we have discarded a total (functional) differential ($\int_{\Sigma} \text{Tr}(\lambda\delta\tilde{A}^\mu)d\sigma_\mu$).

The expression (4.28) tells us directly the actual conjugate couples of basic ‘‘coordinates’’ and ‘‘momenta’’ to be quantized. It is apparent now that the components of \tilde{A}_μ perpendicular to Σ and λ (in space-time and group directions, respectively) have the gauge group parameters U themselves as conjugate coordinates.

Let us finish this Section by writing the explicit form of the classical Hamiltonian, considered as the Noether charge associated with the time-translation symmetry of (4.16) in the simplest way:

$$H = \frac{1}{2} \int d^3x \text{Tr}\{(E_i^2 + B_i^2) + m^2([\theta_0, \Lambda]^2 + [\theta^i - A^i, \Lambda]^2)\}, \quad (4.29)$$

where E and B are defined in the standard way in terms of the curvature tensor F .

To arrive at this expression we have chosen the vector n in the form $(1, 0, 0, 0)$ and the gauge-invariance freedom has been used to fix $A_0 = 0$, required to make the Hamiltonian positive-definite.

4.4.2 Group Approach to Quantization of Massive Yang-Mills Fields

4.4.2.1 The Massive Yang-Mills Symmetry Group and Noether Invariants

We are now in conditions to write a group law, G_{MYM} , providing the solution manifold of our Massive Yang-Mills system, as a co-adjoint orbit, along with the symplectic potential $\Theta_{MYM}^{G/H}$, as the $U(1)$ -component of the left-invariant 1-form (except for a total

differential). We write for such a group,

$$\begin{aligned}
U''(x) &= U'(x)U(x) \\
\theta''_\mu(x)n^\mu &= U'(x)\theta_\mu(x)n^\mu U'^\dagger(x) + \theta'_\mu(x)n^\mu, \\
A''_\mu(x) &= U'(x)A_\mu(x)U'^\dagger(x) + A'_\mu(x), \\
F''_{\mu\nu}(x) &= U'(x)F_{\mu\nu}(x)U'^\dagger(x) + F'_{\mu\nu}(x), \tag{4.30} \\
\zeta'' &= \zeta'\zeta \exp\left(i \int_\Sigma d\sigma^\mu(x) J_\mu(U', A', F'; U, A, F)\right), \\
J_\mu &= J_\mu^{\text{YM}} + J_\mu^\sigma, \\
J_\mu^{\text{YM}} &= \frac{1}{2} \text{Tr} \left((A'^\nu - \theta'^\nu) U' (F_{\mu\nu} - F_{\nu\mu}) U'^\dagger - (F'_{\mu\nu} - F'_{\nu\mu}) U' (A^\nu - \theta^\nu) U'^\dagger \right), \\
J_\mu^\sigma &= m \text{Tr} \left(\lambda (U' (A_\mu - \theta_\mu) U'^\dagger - (A_\mu - \theta_\mu)) \right).
\end{aligned}$$

where all fields are assumed to be defined on the Cauchy surface Σ , so that, the time translation can not be directly implemented, in contrast with the case of free fields [102]. However, we shall construct an explicit Hamiltonian operator to account for the time evolution on the quantum states (see below)³. This group constitutes the minimal symmetry necessary to reproduce the solution manifold associated with the classical Lagrangian (4.16). Apart from the possible inclusion of the Poincaré subgroup (see below), this group could also be added with those local transformations parametrized by parameters $\varphi^a(x)$ not subjected to satisfy the equations of motion, but they are gauge in the strict sense so that they preserve the solution manifold pointwise.

Before proceeding further in any actual computation according to the general GAQ scheme, let us comment on several facts. Firstly, the objects A_μ^a and θ_μ^a behave exactly in the same manner under the local group $G(M)$ and θ_i^a (spacial components) are particular cases of A_i^a , that is, the cases in which the Yang-Mills potentials are pure gauge. Secondly, there are some extra unexpected parameters A_0^a , with respect to the massless case, which are the zero (time) components of those vector potentials living on the orbit G/H . They also behave as the time components of θ_μ^a but they never are pure gauge, as they fix the initial values of the derivatives of the fields φ^a on the Cauchy surface Σ . We then require non-trivial (non-gauge) four components A_μ^a for each index a of the orbit of G . We accordingly add a time component for the “electric” field E^a , $E_i^a = F_{0i}^a$, namely $E_0^a = m^2 \text{Tr}(T^a \Lambda)$,

³The coboundary piece J_μ^σ in (4.31) does not exhaust all possibilities of encoding dynamical content in the group. In fact, the coboundary current $J_\mu^{\prime\sigma} = i\kappa n_\mu \text{Tr}(\lambda(\ln(U'U) - \ln(U') - \ln(U)))$ was considered in ([19]), as a preliminary attempt to give mass to vector bosons, but we shall not pursue this possibility any further here.

constituting, somehow, a new component, “ F_{00} ”, of the curvature $F_{\mu\nu}^a$. From the mathematical point of view, the appearance of this non-conventional field degrees of freedom are a consequence of the piece J^σ of the current in (4.31).

From this group law (4.31), the generators of the left group action (the right-invariant vector fields) can be written:

$$\begin{aligned}
\tilde{X}_{\varphi^a(x)}^R &= X_{\varphi^a(x)}^{R(G)} - C_{ab}^c \left(\theta_\nu^b(x) n^\nu \frac{\delta}{\delta \theta_\mu^c(x) n^\mu} + A_\nu^b(x) \frac{\delta}{\delta A_\nu^c(x)} - C_{ab}^c F_{\mu\nu}^b(x) \frac{\delta}{\delta F_{\mu\nu}^c(x)} \right) \\
&+ \left(-C_{ab}^c m(A_\nu^b(x) - \theta_\mu^b(x)) \lambda_c + \frac{1}{2} \partial^\nu (F_{a\mu\nu} - F_{a\nu\mu}) \right) n^\nu(x) \Xi \\
\tilde{X}_{\theta_\mu^a(x) n^\mu}^R &= \frac{\delta}{\delta \theta_\mu^a(x) n^\mu} \\
\tilde{X}_{A_\mu^a(x)}^R &= \frac{\delta}{\delta A_\mu^a(x)} - \frac{1}{2} (F_a^{\mu\nu}(x) - F_a^{\nu\mu}(x)) n_\nu(x) \Xi \\
\tilde{X}_{F_{\mu\nu}^a(x)}^R &= \frac{\delta}{\delta F_{\mu\nu}^a(x)} - \frac{1}{2} \left((A_a^\mu(x) - \theta_a^\mu(x)) n^\nu(x) - (A_a^\nu(x) - \theta_a^\nu(x)) n^\mu \right) \Xi \\
\tilde{X}_\zeta^R &= \text{Re} \left(i \zeta \frac{\partial}{\partial \zeta} \right) \equiv \Xi.
\end{aligned} \tag{4.31}$$

The corresponding non-null (equal-time) Lie bracket are:

$$\begin{aligned}
\left[\left[\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{\varphi^b(y)}^R \right] \right] &= -C_{ab}^c \delta(x-y) \tilde{X}_{\varphi^c(x)}^R \\
\left[\left[\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{\theta_\mu^b(y) n^\mu}^R \right] \right] &= -C_{ab}^c \delta(x-y) \tilde{X}_{\theta_\mu^c(x) n^\mu}^R + m C_{ab}^c \lambda_c \delta(x-y) \Xi \\
\left[\left[\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{A_\mu^b(y)}^R \right] \right] &= -C_{ab}^c \delta(x-y) \tilde{X}_{A_\mu^c(x)}^R - m \delta_0^\mu C_{ab}^c \lambda_c \delta(x-y) \Xi \\
\left[\left[\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{E_j^b(y)}^R \right] \right] &= -C_{ab}^c \delta(x-y) \tilde{X}_{E_j^c(x)}^R - \delta_a^b \partial_j^x \delta(x-y) \Xi \\
\left[\left[\tilde{X}_{A_j^a(x)}^R, \tilde{X}_{E_k^b(y)}^R \right] \right] &= \delta^{jk} \delta_{ab} \delta(x-y) \Xi,
\end{aligned} \tag{4.32}$$

where double bracket indicates the commutator in the Lie algebra G_{MYM} (as opposed to the commutator in the Lie algebra of G). In the same way we derive the left-invariant vector

fields:

$$\begin{aligned}
\tilde{X}_{\varphi(x)}^L &= X_{\varphi(x)}^{L(G)} + \frac{1}{2} \partial^\mu (U^\dagger (F_{\mu\nu} - F_{\nu\mu}) U) n^\mu \Xi \\
\tilde{X}_{\theta_\mu(x) n^\mu}^L &= U \frac{\delta}{\delta \theta_\mu(x) n^\mu} U^\dagger - m (U^\dagger \lambda U - \lambda) \Xi \\
\tilde{X}_{A_\mu(x)}^L &= U \frac{\delta}{\delta A_\mu} U^\dagger + \left(m (U^\dagger \lambda U - \lambda) \eta^{\mu\nu} - \frac{1}{2} U^\dagger (F^{\nu\mu} - F^{\mu\nu}) U \right) n_\nu \Xi \\
\tilde{X}_{F_{\mu\nu}(x)}^L &= U \frac{\delta}{\delta F_{\mu\nu}(x)} U^\dagger + \frac{1}{2} [U^\dagger (A^\nu - \theta^\nu) U n^\mu - U^\dagger (A^\mu - \theta^\mu) U n^\nu] \Xi \\
\tilde{X}_\zeta^L &= \text{Re} \left(i \zeta \frac{\partial}{\partial \zeta} \right) \Xi. \tag{4.33}
\end{aligned}$$

Directly from the group law or by duality on (4.33) the left-invariant 1-form in the ζ -direction, Θ_{MYM}^G , can be computed (the tensor F will be considered anti-symmetric from now on):

$$\begin{aligned}
\Theta_{MYM}^G &= \int_\Sigma d\sigma^\nu \text{Tr} \left(F_{\nu\mu} \delta A^\mu - \frac{1}{2} (\delta F_{\nu\sigma} - \delta F_{\sigma\nu}) (A^\sigma - \theta^\sigma) + m (\Lambda - \lambda) \delta (A_\nu - \theta_\nu) \right. \\
&\quad \left. + F_{\nu\mu} [\theta^\mu, -i \delta U U^\dagger] + \partial^\mu F_{\nu\mu} (-i \delta U U^\dagger) \right) + \frac{d\zeta}{i\zeta} \\
&= \int_\Sigma d\sigma^\nu \text{Tr} \left((F_{\mu\nu} \delta (A^\mu - \theta^\mu) - (A^\mu - \theta^\mu) \delta F_{\mu\nu}) + m (\Lambda - \lambda) \delta (A_\nu - \theta_\nu) \right) + \frac{d\zeta}{i\zeta}. \tag{4.34}
\end{aligned}$$

and from it the Noether invariants ($I = i_{X^R} \Theta_{MYM}^G$):

$$\begin{aligned}
I_{A^\mu} &= (F_{\nu\mu} + m (\Lambda - \lambda) \eta_{\nu\mu}) n^\nu \\
I_{F^{\mu\nu}} &= (A_\mu - \theta_\mu) n_\nu - (A_\nu - \theta_\nu) n_\mu \\
I_\varphi &= \partial^\nu F_{\mu\nu} n^\mu + [(A^\mu - \theta^\mu), m \Lambda \eta_{\mu\nu} + F_{\nu\mu}] n^\nu \\
I_{\theta^\mu n_\mu} &= m (\Lambda - \lambda), \tag{4.35}
\end{aligned}$$

For the particular, though standard, choice of the Cauchy surface $n = (1, 0, 0, 0)$ they ac-

quire the expressions:

$$\begin{aligned}
I_{A^0} &= m(\Lambda - \lambda) \equiv m\mathbb{E}_0 - m\lambda \\
I_{A^i} &= F_{0i} \equiv \mathbb{E}_i \\
I_{F^{0i}} &= -(A_i - \theta_i) \equiv -\mathbb{A}_i \\
I_\varphi &= m[(A^0 - \theta^0), \Lambda] + [(A^i - \theta^i), F_{0i}] + \partial^i F_{0i} \equiv -m\check{\mathbb{L}} - \vec{\nabla} \cdot \vec{\mathbb{E}} + [\mathbb{A}^i, \mathbb{E}_i] \equiv \mathbb{G} \\
I_{\theta^0} &= m(\Lambda - \lambda) = I_{A^0}, \tag{4.36}
\end{aligned}$$

where we have denoted $\check{\mathbb{L}} \equiv [(\theta^0 - A^0), \Lambda]$ the minimally coupled version of the intrinsic internal “spin” \mathbb{L} , which already appeared in the sigma sector. Note that the expression of \mathbb{G} would look very much like $D_\mu \mathbb{E}^\mu = \partial_\mu \mathbb{E}^\mu + [\mathbb{A}_\mu, \mathbb{E}^\mu]$ had we identified $-\mathbb{L}$ with some sort of formal time derivative “ $\partial^0 \mathbb{E}^0$ ” and A_0 with “ \mathbb{A}_0 ”, where the quotation marks mean that neither “ $\partial^0 \mathbb{E}^0$ ” is a time derivative nor A_0 itself is a Noether invariant.

We should observe that the Noether invariants I_{θ^0} and I_{A^0} coincide, so that the group parameter $A^0 - \theta^0$ is actually a gauge parameter in the strict sense (the corresponding subgroup leaves the solution manifold invariant pointwise). Notice also that the Noether invariant I_{A_0} in the direction of the H subalgebra is a function of the invariants I_{A_0} in the direction of the G/H orbit (the same dependence occurs for I_{θ_0}). Then, the independent parameters in the solution manifold are:

$$(\mathbb{A}_i^a, \mathbb{E}_j^a, \mathbb{E}_0^b, \mathbb{G}^b) \quad a = 1, \dots, \dim G, \quad b = 1, \dots, \text{co-dim } H. \tag{4.37}$$

The 1-form (4.34) naturally comes down to the quotient G/H , which can be identified with the coadjoint orbit of G , reproducing (4.27) up to a total differential. It can also be written in terms of Noether invariants. In fact, we have

$$\Theta_{MYM}^G = \int_\Sigma d\sigma^\nu \text{Tr} \{ (I_{A_\mu} \delta I_{F^{\mu\nu}} - I_{F^{\mu\nu}} \delta I_{A_\mu}) + I_{\theta_\mu n^\mu} \delta ([I_\varphi, I_{A^\nu}]) \} + \frac{d\zeta}{i\zeta}. \tag{4.38}$$

The Noether invariants (4.35) close the standard Poisson bracket along with those corresponding to the new variables E_0^a , which are absent from the conventional theory [38, 27]. The non-zero brackets, choosing $\Sigma = \mathbb{R}^3$ in the time direction (i.e., $d\sigma_\mu \rightarrow d^3x$), are:

$$\begin{aligned}
\{\mathbb{G}_a(\vec{x}), \mathbb{G}_b(\vec{y})\} &= -C_{ab}^c \mathbb{G}_c(\vec{x}) \delta(\vec{x} - \vec{y}), \\
\{\mathbb{A}_j^a(\vec{x}), \mathbb{E}_b^k(\vec{y})\} &= -\delta_j^k \delta_b^a \delta(\vec{x} - \vec{y}), \tag{4.39} \\
\{\mathbb{G}_a(\vec{x}), \mathbb{A}_j^b(\vec{y})\} &= -C_{ac}^b \mathbb{A}_j^c(\vec{x}) \delta(\vec{x} - \vec{y}) + \delta_a^b \partial_j^x \delta(\vec{x} - \vec{y}), \\
\{\mathbb{G}_a(\vec{x}), \mathbb{E}_b^\mu(\vec{y})\} &= -C_{ab}^c \mathbb{E}_c^\mu(\vec{x}) \delta(\vec{x} - \vec{y}) + m \delta_0^\mu C_{ab}^c \lambda_c \delta(\vec{x} - \vec{y}).
\end{aligned}$$

Even though we have not included space-time translations (nor Lorentz transformations) explicitly in the group law, this Poisson algebra can be added with a quartic function of the Noether invariants, \mathbb{H} , constituting the classical Hamiltonian in our group-theoretical scheme. In fact, the function

$$\mathbb{H} = \frac{1}{2} \int d^3x \text{Tr}\{(\vec{\mathbb{E}}^2 + \vec{\mathbb{B}}^2) + m^2 \check{\mathbb{L}}^2 + m^2(\vec{\nabla} \mathbb{E}_0 - [\vec{\mathbb{A}}, \mathbb{E}_0])^2\} \quad (4.40)$$

where we have designed by $\check{\mathbb{L}}$ the “internal” part of the \mathbb{G} (also a constant of motion), that is,

$$m \check{\mathbb{L}}^a \equiv -\mathbb{G}^a + [\mathbb{A}^i, \mathbb{E}_i]^a + \partial^i \mathbb{E}_i^a, \quad (4.41)$$

recovers the classical equations of motion, using the Poisson brackets, from the solution manifold⁴.

It should be remarked that in comparing the expressions of the (classical) Hamiltonians (4.29) and (4.40), both written in terms of variables in the corresponding solution manifolds, we must carry out the correspondence

$$E_i \leftrightarrow \mathbb{E}_i, \quad \Lambda \leftrightarrow \mathbb{E}_0, \quad A_i \leftrightarrow \mathbb{A}_i$$

and realize that $[\theta_i, \Lambda] = \partial_i \Lambda$.

4.4.2.2 The Quantum Representation

According to the general scheme of GAQ we start from complex $U(1)$ -functions Ψ on the entire centrally-extended group to be represented, that is, functions which are homogeneous of degree one on the parametre $\zeta \in U(1)$. In order to obtain an irreducible representation these functions must be restricted by the Polarization condition established by means of a polarization subgroup G_P of the finite left action. A look at the Lie algebra commutators of our symmetry group reveals that the actual Polarization subgroup (see (2.10)) is constituted by the following elements:

$$g_P = (U_H, \theta^\mu n_\mu, \check{A}, F=0, \zeta=1), \quad \check{A} \equiv (A - \theta), \quad (4.42)$$

which act on the original complex functions $\Psi(g') = \Psi(U', \theta'_\mu n^\mu, A'_\nu, F'_{\mu\nu}, \zeta')$ from the right: $\Psi(g') \rightarrow \Psi(g' g_P)$.

The key point in searching for the appropriate form of the wave functions, invariant under the Polarization subgroup, is to notice the factor which appears in front of the wave function as a consequence of the cocycle in the composition law of the $U(1)$ argument (Ψ is homogeneous of degree one on it). This factor must be canceled out by some $U(1)$ term factorizing an arbitrary function Φ of given arguments. It is easy to see that the following wave function ψ is the general solution to the polarization equation (2.10):

$$\psi = \zeta e^{i \int_\Sigma d\sigma^\mu \text{Tr}(m \lambda (U^\dagger \check{A}_\mu U - \check{A}_\mu) + \frac{1}{2} \check{A}^\nu F_{\mu\nu})} \Phi((\Lambda - \lambda), F) \quad (4.43)$$

⁴Remind that Poisson brackets are defined between Noether invariants only and that if Q is the Noether invariant associated with a parameter q evolving in time with the Hamiltonian H , the Poisson bracket Q, H is the Noether invariant \dot{Q} associated with the time derivative \dot{q} of q (obviously, \dot{Q} is not a time derivative of Q , since it is a Noether invariant).

where Φ is an arbitrary function of its arguments. In fact, choosing $d\sigma^\mu$ as d^3x , without loss of generality,

$$\begin{aligned} \psi(g' g_p) &= \zeta' e^{i \int d^3x \text{Tr}[m\lambda((U' U_H)^\dagger (U' \check{A}_0 U'^\dagger + \check{A}'_0)(U' U_H) - U' \check{A}_0 U'^\dagger - \check{A}'_0) + \frac{1}{2}(U' \check{A}^\nu U'^\dagger + \check{A}'^\nu) F'_{0\nu}]} \\ &\times e^{i \int d^3x \text{Tr}[m\lambda(U' \check{A}_0 U'^\dagger - \check{A}_0) - \frac{1}{2} F'_{0\nu} U' \check{A}^\nu U'^\dagger]} \Phi(U' U_H \lambda (U' U_H)^\dagger - U_H \lambda U_H^\dagger, F') \\ &= \psi(g') \end{aligned} \quad (4.44)$$

For the usual choice of Σ , the arbitrary part of ψ can be written in terms of the variables (4.37). $\Phi = \Phi(\mathbb{E}_\mu^a)$, if we adopt the convention that $\mathbb{E}_0^c = 0$, $\forall c$ running on H . That is, we arrive at the (generalized) “electric field representation”.

The action of the right-invariant vector fields preserve the space of polarized wave functions, due to the commutativity of the left and right actions as already stated, so that it is possible to define an action of them on the arbitrary factor Φ in the wave functions. It is not difficult to demonstrate that on this space of functions the quantum operators acquire the following expression:

$$\begin{aligned} \hat{E}_a^\mu \Phi &\equiv i \zeta^{-1} e^{-i \int_\Sigma d\sigma^\mu \text{Tr}(m\lambda(U^\dagger \check{A}_\mu U - \check{A}_\mu) + \frac{1}{2} \check{A}^\nu F_{\mu\nu})} \tilde{X}_{A_\mu^a}^R \psi = \mathbb{E}_a^\mu \Phi \\ \hat{A}_a^i \Phi &\equiv -i \zeta^{-1} e^{-i \int_\Sigma d\sigma^\mu \text{Tr}(m\lambda(U^\dagger \check{A}_\mu U - \check{A}_\mu) + \frac{1}{2} \check{A}^\nu F_{\mu\nu})} \tilde{X}_{F_{0i}^a}^R \psi = -i \frac{\delta}{\delta \mathbb{E}_a^i} \Phi \\ \hat{G}_a \Phi &\equiv -i \zeta^{-1} e^{-i \int_\Sigma d\sigma^\mu \text{Tr}(m\lambda(U^\dagger \check{A}_\mu U - \check{A}_\mu) + \frac{1}{2} \check{A}^\nu F_{\mu\nu})} \tilde{X}_{\varphi^a}^R \psi \\ &= \left(\vec{\nabla} \cdot \vec{\mathbb{E}}_a + C_{ab}^c (\mathbb{E}_0^b \frac{\delta}{\delta \mathbb{E}_0^c} - \vec{\mathbb{E}}^b \cdot \frac{\delta}{\delta \vec{\mathbb{E}}^c}) \right) \Phi. \end{aligned} \quad (4.45)$$

The last expression accounts for the non-Abelian “Gauss law” if the constraint condition $\hat{G}_a \Phi(\mathbb{E}) = 0$ is required.

On the quantum representation space we can construct the Hamiltonian operator \hat{H} that represents, without ambiguity, the classical Hamiltonian (4.40):

$$\hat{H}\Phi = \frac{1}{2} \int d^3x \{ (\hat{E}_i^2 + \hat{B}_i^2) + m^2 \hat{L}^2 + m^2 (\partial^i \hat{E}_0 - [\hat{A}_i, \hat{E}_0])^2 \} \Phi. \quad (4.46)$$

It is of remarkable relevance the fact that this operator preserves the Hilbert space of quantum states.

It should be stressed that the central term proportional to λ_c in the last commutator of (4.32) could also be considered as a remnant of some sort of “symmetry breaking” in the sense that it can be hidden into a redefinition of \hat{E}_a^0 ,

$$\hat{E}_a^0 \rightarrow \hat{E}_a'^0 \equiv \hat{E}_a^0 + m\lambda_a, \quad (4.47)$$

which now acquires a non-zero vacuum expectation value proportional to the mass $m^2 \lambda_a$, that is:

$$\langle 0 | \hat{E}_a^0 | 0 \rangle = 0 \longrightarrow \langle 0 | \hat{E}_a'^0 | 0 \rangle = m\lambda_a. \quad (4.48)$$

4.5 The $SU(2) \times U(1)$ group and the Standard Model

Now we shall denote by $B_\mu = B_\mu^a T_a$, $a = 1, \dots, 4$, the $SU(2) \times U(1)$ Lie-algebra valued vector potential, keeping A_μ for the electromagnetic potential, as usual. The new generator T_4 ($\equiv \frac{1}{2}Y$, the halved hypercharge) corresponds to the direct factor $U(1)$.

The key point in this section consists in combining the construction above for $G = SU(2)$ with the traditional Stueckelberg model for a selected $H^\perp = U(1)$. However, this time we shall choose λ in the electric charge (mixed) direction

$$\lambda \propto Q \equiv T_3 + T_4, \quad (4.49)$$

according to the usual Gell-Mann-Nishijima relation, and we shall choose H^\perp in the orthogonal direction:

$$Q^\perp = T_3 - T_4, \quad (4.50)$$

in the sense that $\text{Tr}(QQ^\perp) = 0$. This way we shall provide mass to three vector bosons, say $W_\mu^\pm \propto B_\mu^1 \pm iB_\mu^2$ and $Z_\mu \propto \text{Tr}(Q^\perp B_\mu)$, out of the original four vector potentials B_μ^a , $a = 1, \dots, 4$, leaving the electromagnetic potential A_μ massless. In fact, the Standard Model Lagrangian for the Yang-Mills sector will be

$$\begin{aligned} \mathcal{L}_{\text{MYM}}^{\text{SM}} &= -\frac{1}{4}\text{Tr}(F_{\mu\nu})^2 + \frac{1}{2}m^2\text{Tr}([\theta_\mu - B_\mu, UQU^\dagger]^2) + \frac{1}{2}m'^2\text{Tr}((\theta_\mu - B_\mu)UQ^\perp U^\dagger)^2 \\ &= -\frac{1}{4}\text{Tr}(F_{\mu\nu})^2 + \frac{1}{2}m^2\text{Tr}([\tilde{B}_\mu, Q]^2) + \frac{1}{2}m'^2\text{Tr}((\tilde{B}_\mu Q^\perp)^2) \end{aligned} \quad (4.51)$$

$$\equiv -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + m_W^2 \tilde{W}_\mu^+ \tilde{W}^{\mu-} + \frac{1}{2}m_Z^2 \tilde{Z}_\mu \tilde{Z}^\mu, \quad (4.52)$$

where \tilde{B}_μ is related to B_μ in a way similar to that of Eq. (4.17). This Lagrangian reproduces the Yang-Mills sector of the Standard Model for electroweak interactions when we introduce the usual coupling constants g, g', e according to $\tilde{B}_\mu^3 \equiv g \mathcal{B}_\mu^3$, $\tilde{B}_\mu^4 \equiv g' \mathcal{B}_\mu^4$, $\tilde{Z}_\mu \equiv \frac{gg'}{e} \mathcal{Z}_\mu$. Writing \mathcal{Z}_μ in terms of \mathcal{B}_μ^3 and \mathcal{B}_μ^4 , we have:

$$\mathcal{Z}_\mu = \frac{e}{gg'} \tilde{Z}_\mu = \frac{e}{gg'} (g \mathcal{B}_\mu^3 - g' \mathcal{B}_\mu^4) \equiv \cos(\vartheta_W) \mathcal{B}_\mu^3 - \sin(\vartheta_W) \mathcal{B}_\mu^4, \quad (4.53)$$

which, together with the orthogonal relation

$$\mathcal{A}_\mu \equiv \sin(\vartheta_W) \mathcal{B}_\mu^3 + \cos(\vartheta_W) \mathcal{B}_\mu^4, \quad (4.54)$$

(the electromagnetic vector potential) defines the usual Weinberg rotation of angle ϑ_W .

4.5.1 Giving mass to fermionic matter

The introduction of mass for fermionic matter can be accomplished by a nontrivial mixing between space-time and internal symmetries. Although the general setting of this

symmetry mixing is rather ambitious, here we shall consider the consequences of the simplest, nontrivial, mixing between the Poincaré group \mathcal{P} and the electromagnetic gauge subgroup $H = U(1)_Q$, which has been widely developed in [64] and references therein. A more general symmetry mixing scheme, involving conformal symmetry and larger internal symmetries, is under consideration [105].

To be precise, we propose a mass-generating mechanism associated with a non-trivial mixing of the Poincaré group and $SU(2) \times U(1)$. This mixing takes place through a linear combination $P'_0 \equiv P_0 + \kappa Q$ between the time translation generator P_0 and Q , in much the same way the generator Q had to be found as a linear combination of T_3 and T_4 . The spirit of the redefinition P'_0 is the same as the shifting (4.47) [with $\lambda \propto Q$], ultimately responsible for the mass m_W . In fact, with the new mass operator

$$M'^2 \equiv P_0'^2 - \vec{P}'^2 \quad (4.55)$$

the mass shell condition for fermionic fields ψ becomes

$$M^2 \psi = (P_0^2 - \vec{P}^2) \psi = m_0^2 \psi \rightarrow M'^2 \psi = (m_0^2 + 2\kappa P_0 Q + \kappa^2 Q^2) \psi. \quad (4.56)$$

At the rest frame we have

$$M'^2 \psi = (m_0^2 + 2\kappa m_0 Q + \kappa^2 Q^2) \psi. \quad (4.57)$$

Then, for “originally” massless particles ($m_0 = 0$),

$$M'^2 \psi = \kappa^2 Q^2 \psi, \quad (4.58)$$

so that only charged particles acquire mass. This is in agreement with the fact that there is no elementary fermionic massive particles without electric charge.

Perturbation theory for non-linear sigma models

From an abstract (mathematical) point of view, a Non-Linear Sigma Model (NLSM) consists of a set of coupled scalar fields $\pi^a(x^\mu)$, $a = 1, \dots, N$, in a D -dimensional (Minkowski) spacetime M with coordinates x^μ , $\mu = 0, 1, 2, \dots, D - 1$, and action integral

$$S_\sigma(\pi, \partial_\mu \pi) = \int_M \mathcal{L}(\pi, \partial_\mu \pi) d^D x = \frac{\lambda}{2} \int_M g_{ab}(\pi) \partial^\mu \pi^a \partial_\mu \pi^b d^D x, \quad (5.1)$$

where $\partial^\mu = \eta^{\mu\nu} \partial_\nu$, $\partial_\nu = \partial / \partial x^\nu$, $\eta = \text{diag}(+, -, \dots, -)$ is the Minkowski metric and λ a coupling constant. The field theory (5.1) is called the NLSM with metric $g_{ab}(\pi)$ (usually a positive-definite field-dependent matrix). The fields π^a themselves could also be considered as the coordinates of an internal Riemannian (target) manifold Σ with metric g_{ab} . This model proved to be relevant in String Theory where g_{ab} is the Einstein metric and M is a two-dimensional manifold named “worldsheet”. An interesting case for us is that in which Σ is a Lie group manifold G , namely $G = O(N)$, or a quotient (coset) space G/H by a closed subgroup H , namely $H = O(N - 1)$ (see [45] for $G = U(N)$ and its cosets G/H : complex projective, Grassmann and flag manifolds).

Apart from String Theory, the NLSM is related to a great number of physical systems (see e.g. [100] for a review). It was originally introduced to describe pion dynamics in the theory of strong nuclear interactions. Also, some particular two-dimensional $O(N)$ -invariant NLSM are used in connection to antiferromagnetic spin chains, quantum Hall effect and superfluid helium-3. At a more fundamental level, NLSM describes the dynamics of Goldstone bosons in spontaneously broken field theories like the Standard Model of electro-weak interactions. In the previous chapter we have proposed a Higgs-less mechanism to provide mass to the electro-weak gauge vector bosons W_\pm and Z through a coupling to a $U(2)$ -invariant NLSM *à la* Stueckelberg. Actually, according to the widely named “Equivalence Theorem” [97, 98], a very heavy Higgs particle can be eliminated from the broken symmetry programme in favor of non-linear σ -like Goldstone bosons, so that the actual computation of Feynman diagrams involving the longitudinal polarizations of the (massive) vector bosons in electroweak interactions can be resolved in terms of the corresponding diagrams among those scalar fields. Unfortunately, the use of a NLSM Lagrangian has led to an apparent insoluble dichotomy unitarity-renormalizability [62, 56, 99, 48, 59] (see also the review [46] and references therein). Canonical perturba-

tion theory proceeds from the action (5.1) by expanding $g_{ab}(\pi) = \delta_{ab} + O(\pi^2)$ and perturbing around massless fields fulfilling $\partial_\mu \partial^\mu \pi^a = 0$. However, this perturbation scheme is subject to criticism. On the one hand, massless solutions do not exhaust the whole solution manifold, as other (soliton, instanton, skyrmion) solutions are known to exist [106]. On the other hand, the non-trivial (non-flat) geometry and topology of the target manifold Σ and its possible symmetries are not being taken into account or properly exploited. Regarding the last issue, references like [66] tackled the perturbation theory for NLSM in terms of left- G -invariant quantities $L_\mu(x) = g^{-1}(x)\partial_\mu g(x)$, $g \in G$, which do not depend on the parametrization of G .

As in Reference [103], we think that the trouble that canonical quantization faces in dealing with systems bearing non-trivial topology can be traced back to the ‘‘tangent space’’ approximation imposed at the very beginning of the (canonical) quantization program. Already in the simple case of ‘‘free’’ particles moving on spheres, a proper quantization requires the replacement of canonical commutators with the Lie-algebra commutators of the Euclidean group [103, 43]. We shall pursue this idea and construct a perturbation theory adapted to non-canonical (namely, Euclidean) commutation relations for the particular case of $G = O(N + 1)$ invariant NLSM with $\Sigma = S^N = O(N + 1)/O(N)$ the N -dimensional sphere. The discretization of the corresponding equations of motion provides a mechanical picture of the $O(N)$ -invariant NLSM as a $(D - 1)$ -dimensional lattice model of coupled rotators connected by springs (see later on Sec. 5.1). Actually, this equivalence has already been considered in, for instance, [107, 108] who used the so-called ‘‘coupled cluster method’’ to approach this problem. Our aim here is to explore the NLSM in a different regime from the usual (relativistic) one, by using non-canonical basic commutation relations adapted to the underlying $O(N)$ symmetry of the system.

5.1 $O(N + 1)$ -Invariant NLSM

The $O(N + 1)$ -invariant NLSM Lagrangian in (5.1) can be obtained from the quadratic one

$$\mathcal{L}(\vec{\phi}, \partial_\mu \vec{\phi}) = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}, \quad \vec{\phi} = (\phi^1, \dots, \phi^{N+1}) \in \mathbb{R}^{N+1}, \quad (5.2)$$

with the constraint $\vec{\phi}^2 = \rho^2 = \text{constant}$. A NLSM action of type (5.1) can be recovered from this Lagrangian by eliminating ϕ^{N+1} in terms of $\vec{\pi} = (\phi^1, \dots, \phi^N)$ or its stereographic projection on \mathbb{R}^N . Here we shall work with $\vec{\phi}$ and keep in mind the constraint $\vec{\phi}^2 = \rho^2$. Using Lagrange multipliers, the Euler-Lagrange equations of motion can be cast in the form:

$$\square \vec{\phi} = \frac{\square \vec{\phi} \cdot \vec{\phi}}{\vec{\phi}^2} \vec{\phi}, \quad \vec{\phi}^2 = \rho^2, \quad (5.3)$$

where $\square = \partial_\mu \partial^\mu$ denotes the d’Alembertian or wave operator. For $N = 3, D = 2$, extra Wess-Zumino-Novikov-Witten terms can be added to the Lagrangian (5.2) so that the model is known to be integrable since one is able to find an infinite number of conserved quantities closing a Kac-Moody Lie algebra (see e.g. [100]).

Let us briefly remind how the NLSM above also arises from a ϕ^4 -theory by “freezing out” the Higgs field degree of freedom (as in the above-mentioned Equivalence Theorem). Actually, the term “sigma” makes reference to the original model for an effective theory of the meson part of the low-energy nuclear theory. The Lagrangian (5.2) is modified by a Higgs potential

$$\mathcal{L}_g = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \frac{g}{4} (\vec{\phi}^2 - \rho^2)^2, \quad (5.4)$$

with g a positive constant. It is customary to write

$$\phi^{N+1} = \rho + \sigma, \quad \phi^a = \pi^a, \quad a = 1, \dots, N, \quad (5.5)$$

for small perturbations around $\vec{\phi}_{(0)} = (0, \dots, \rho)$. The Lagrangian (5.4) acquires then the following form in terms of $(\vec{\pi}, \sigma)$:

$$\mathcal{L}_g = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{m_\sigma^2 c^2}{2} \sigma^2 + \dots, \quad (5.6)$$

which states that the σ -meson (Higgs field) has mass $m_\sigma = \sqrt{2g\rho}/c$ whereas the π -mesons (pions) remain massless. In fact, in the quantum theory, π^a describe Goldstone bosons associated with the spontaneous breakdown from the $O(N+1)$ to the $O(N)$ symmetry for the choice of vacuum $\langle 0 | \phi^j | 0 \rangle = \rho \delta_{j, N+1}$.

The original NLSM Lagrangian (5.2) can be obtained from (5.4) by taking the limit $g \rightarrow \infty$ and imposing $\vec{\phi}^2 = \rho^2$ in order to keep the Lagrangian finite except for an irrelevant c-number term. This corresponds to $m_\sigma \rightarrow \infty$ so that the Higgs field degree of freedom has been frozen (something physically reasonable since it has not been experimentally observed yet). Note that, even for large g , we could always keep m_σ finite by taking the vacuum expectation value ρ small. Actually, we are interested in this regime in this article.

However, one should be very cautious in taking this limit, since we are dramatically changing the topology of the field configuration space. One can not guarantee in principle that the procedure of *perturbing* commutes with that of *constraining*. In this article we pursue the alternative strategy of “constraining and then perturbing”, instead of the previous scheme of “perturbing and then constraining”. Nowadays it is widely known that *constraining* does not actually commute (in general) with *quantizing* (see e.g. [109, 110, 111, 18] for discussions on non-equivalent quantizations of systems with non-trivial configuration spaces).

Let us restrict ourselves, for the sake of simplicity, to the $N = 2$ case. The equations (5.3) can also be obtained as Hamiltonian equations of motion

$$\dot{\vec{\phi}} = \frac{\partial \vec{\phi}}{\partial t} \equiv \{\vec{\phi}, H\}, \quad \ddot{\vec{\phi}} = \frac{\partial^2 \vec{\phi}}{\partial t^2} \equiv \{\dot{\vec{\phi}}, H\} = \{\{\vec{\phi}, H\}, H\}, \quad (5.7)$$

for the Hamiltonian

$$H = \frac{1}{2} \int d^{D-1}x \left(\frac{\vec{L}^2(x)}{\rho^2} + c^2 (\vec{\nabla} \vec{\phi}(x))^2 \right), \quad (5.8)$$

and the basic equal-time Euclidean (non-canonical) Poisson brackets

$$\{L^i(x), L^j(y)\} = \epsilon^{ij}{}_k L^k(x) \delta(x-y), \quad \{L^i(x), \phi^j(y)\} = \epsilon^{ij}{}_k \phi^k(x) \delta(x-y), \quad (5.9)$$

where $\vec{L} \equiv \vec{\phi} \wedge \dot{\vec{\phi}}$, $(\vec{\nabla} \vec{\phi})^2 \equiv \partial_j \phi_k \partial^j \phi^k$, ϵ^{ij}_k is the antisymmetric symbol and we have introduced the wave velocity c when setting $x^0 = ct$ for later convenience. Actually, if (5.9) are taken as abstract Poisson brackets, with \vec{L} not necessarily related to $\vec{\phi}$, then the equations (5.7) generalize (5.3) by introducing an extra term

$$\square \vec{\phi} = \frac{\square \vec{\phi} \cdot \vec{\phi}}{\vec{\phi}^2} \vec{\phi} + \frac{\vec{L} \cdot \vec{\phi}}{\vec{\phi}^4} \vec{L} \quad (5.10)$$

which could not vanish when $\vec{L} \cdot \vec{\phi} \neq 0$, a situation which arises when ‘‘magnetic monopoles’’ are present and \vec{L} is not necessarily perpendicular to $\vec{\phi}$. We shall restrict ourselves to the case $C_1 = \vec{L} \cdot \vec{\phi} = 0$, which is compatible with the Poisson brackets (5.9) and the constraint $C_2 = \vec{\phi}^2 = \rho^2$, since both C_1 and C_2 are the natural Casimir operators for the Euclidean group.

Perturbing around $\vec{\phi}_{(0)} = (0, 0, \rho)$ as in (5.5), for fixed $\vec{\phi}^2 = \rho^2$, can be interpreted as a ‘‘group contraction’’, which drastically changes the topology of the system. Indeed, this perturbation theory has sense for $\rho \gg 1$. Making the change (5.5) in the last Poisson bracket of (5.9) and taking the limit $\rho \rightarrow \infty$, keeping $\varphi^{1,2} \equiv \pi^{1,2}/\rho$ finite, we recover the canonical Poisson brackets:

$$\{L^i(x), \varphi^j(y)\} = \epsilon^{ij}_3 \delta(x - y), \quad i, j = 1, 2 \quad (5.11)$$

which state that (φ^1, φ^2) and $(L^2, -L^1)$ are couples of canonically-conjugated variables. Therefore, standard (canonical) perturbation theory has sense for large values of ρ , which loses information about the (compact) topology of the system. As already commented, We are interested in the other regime $\rho \ll 1$.

5.2 Classical non-canonical perturbation theory

A solution of $\square \vec{\phi} = -m^2 \vec{\phi}$, for any constant m , is also a solution of (5.3). However, only for massless fields, $m = 0$, the constraint $\vec{\phi}^2 = \rho^2$ is also satisfied. At least at the quantum level, standard perturbation theory proceeds by considering scattering of massless fields $\vec{\phi}$ [100]. However, at the classical level, we know that there are more solutions of (5.3) than massless solutions. In fact, as showed long time ago in [106], the configuration space of a NLSM breaks up into an (infinite) number of components. Indeed, finite energy requires boundary conditions like (for instance) $\vec{\phi}(x) = (0, 0, \rho)$ as $\|x\| \rightarrow \infty$, which means a one-point compactification of \mathbb{R}^{D-1} by S^{D-1} . Thus, if two fields $\vec{\phi}$ and $\vec{\phi}'$ belong to different homotopical classes $\Pi_{D-1}(S^N)$, then they can not be continuously deformed (evolved) one into the other. In particular, one can find (solitonic) solutions that are not wave packets of massless solutions.

Instead of perturbing around massless solutions, we shall adopt the following splitting of the Hamiltonian (5.8)

$$H = H_0 + V, \quad H_0 = \frac{1}{2} \int d^{D-1}x \frac{\vec{L}^2(x)}{\rho^2}, \quad V = \frac{c^2}{2} \int d^{D-1}x (\vec{\nabla} \vec{\phi}(x))^2, \quad (5.12)$$

and consider V as a perturbation for either small c or $\|\vec{\phi}\| = \rho \ll 1$ (with c arbitrary).

Given an initial condition on a Cauchy hypersurface, $\vec{\phi}(t_0, x) = \vec{\phi}_0(x)$ and $\dot{\vec{\phi}}(t_0, x) = \dot{\vec{\phi}}_0(x)$, the general solution to (5.3) can be formally written as:

$$\vec{\phi}(t, x) = e^{(t-t_0)\{,H\}} \vec{\phi}_0(x) = U(t-t_0) \vec{\phi}_0(x), \quad (5.13)$$

where $\{, H\}$ stands for the Liouvillian operator and $U(t-t_0) = e^{(t-t_0)\{,H\}}$ for the evolution operator. Actually, we can exactly integrate the “free” evolution as:

$$\begin{aligned} \vec{\phi}^{(0)}(t, x) &\equiv e^{(t-t_0)\{,H_0\}} \vec{\phi}_0(x) = U_0(t-t_0) \vec{\phi}_0(x) \\ &= \cos\left(\sqrt{\frac{\vec{L}^2(x)}{\rho^4}}(t-t_0)\right) \vec{\phi}_0(x) + \frac{1}{\sqrt{\frac{\vec{L}^2(x)}{\rho^4}}} \sin\left(\sqrt{\frac{\vec{L}^2(x)}{\rho^4}}(t-t_0)\right) \dot{\vec{\phi}}_0(x). \end{aligned} \quad (5.14)$$

We shall let the wave velocity c to take arbitrary values, as we want our perturbation theory to be valid for relativistic fields too. We have already justified the interesting regime $\rho \ll 1$ (small vacuum expectation value) in which the Higgs mass m_σ would remain finite while $\vec{\phi}^2 \simeq \rho^2$, so that the Higgs field degree of freedom is almost frozen. In order to gain more physical intuition on this limit, let us use the following mechanical picture of coupled small rotators (see Figure 5.1).

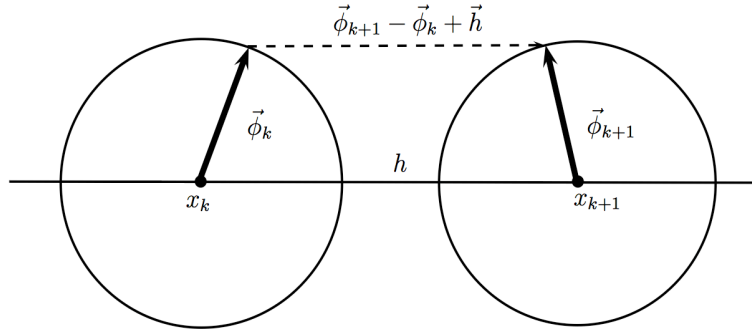


Figure 5.1: Rotators in a lattice coupled by springs

Without loss of generality, we can restrict ourselves to $D = 2$, consider the lattice $x_k = kh, k \in \mathbb{Z}$, for some (small) step h , and write $\vec{\phi}_k(t) = \vec{\phi}(t, x_k)$ for the vector position of the rotator in the place x_k . Rotators are connected by identical springs of constant κ and zero natural length so that the elastic potential energy between two consecutive rotators is

$$V_{k+1,k} = \frac{1}{2} \kappa (\vec{\phi}_{k+1} - \vec{\phi}_k + h(1,0))^2. \quad (5.15)$$

Taking the limit $h \rightarrow 0$, keeping $\kappa h \equiv c^2$ finite, we have that the total elastic potential energy is

$$\sum_{k=-\infty}^{\infty} V_{k+1,k} = \frac{1}{2} c^2 \sum_{k=-\infty}^{\infty} h \frac{(\vec{\phi}_{k+1} - \vec{\phi}_k + h(1,0))^2}{h^2} \rightarrow \frac{1}{2} c^2 \int_{-\infty}^{\infty} dx ((\partial_x \vec{\phi})^2 + 2\partial_x \phi^1). \quad (5.16)$$

which gives the desired result up to a boundary term.

Contrary to the (unconstrained) Klein-Gordon field (as a model of coupled oscillators), the elastic potential energy V can be made arbitrarily small for NLSM fields (as a model of coupled rotators) by taking $\rho \ll 1$, even in rigid media (c arbitrary). In other words, unlike a NLSM, a Klein-Gordon field could never be seen as an infinite set of weakly coupled oscillators unless inside soft media ($c \ll 1$) where it takes a long time for the wave to propagate. That is, here we have the vacuum expectation value ρ as an extra perturbation parameter to play with.

Although Dyson series are conventionally designed for quantum perturbation theory, we shall briefly remind the subject here in a classical setting. Dyson series takes advantage of the exact solvability of H_0 , with exact solution (5.14), to provide a perturbation series in V . The evolution operator (5.13) is decomposed as:

$$U(t, t_0) = U_0(t) \underbrace{U_0(-t)U(t-t_0)U_0(t_0)}_{U_I(t, t_0)} U_0(-t_0),$$

where $U_I(t, t_0)$ is the evolution operator in the interaction image. Let us set $t_0 = 0$ for simplicity. After a little bit of algebra, one can see that

$$\frac{\partial}{\partial t} U_I(t) = \hat{V}(t)U_I(t), \quad \hat{V}(t) \equiv U_0(t)\{, V(\vec{\phi})\}U_0(-t) = \{, V(\vec{\phi}^{(0)}(-t))\}, \quad (5.17)$$

with $\vec{\phi}^{(0)}(t)$ given by (5.14) (note the time inversion). This formula can be recursively integrated as:

$$U_I(t) = I + \int_0^t d\tau \hat{V}(\tau) + \int_0^t d\tau \int_0^\tau d\tau' \hat{V}(\tau)\hat{V}(\tau') + \dots \quad (5.18)$$

In order to test the perturbation procedure, let us consider the exactly solvable case $N = 1, D = 2$. On the one hand, if we parametrize the field $\vec{\phi} = (\phi^1, \phi^2)$ in polar coordinates $\phi = \rho e^{i\theta}$, then (5.3) reduces to a massless Klein-Gordon equation for $\square\theta = 0$. On the other hand, we can compute order by order:

$$\phi(t, x) = U(t)\phi_0(x) = U_0(t)U_I(t)\phi_0(x) = U_0(t)\phi^{(I)}(t, x), \quad (5.19)$$

with

$$\phi^{(I)}(t, x) = U_I(t)\phi_0(x) = \phi_0(x) + \int_0^t d\tau \{\phi_0(x), V(\phi^{(0)}(-\tau))\} + \dots, \quad (5.20)$$

where $V(\phi) = \frac{c^2}{2} \int_{-\infty}^{\infty} dx \partial_x \phi \partial_x \bar{\phi}$ and Poisson brackets are computed at $\tau = 0$. Taking into account that

$$\phi^{(0)}(\tau, x) = U_0(\tau)\phi_0(x) = e^{i\tau L(x)/\rho^2} \phi_0(x) \quad (5.21)$$

with $L = \phi^1 \dot{\phi}^2 - \phi^2 \dot{\phi}^1 = \text{Im}(\bar{\phi} \dot{\phi})$, and that

$$\{L(x), \phi(y)\} = -i\phi(x)\delta(x-y), \quad \{\Psi(L(x)), \phi(y)\} = \Psi'(L(x))\{L(x), \phi(y)\}, \quad (5.22)$$

for any derivable function Ψ of L , we can compute at first order:

$$\begin{aligned}\phi_0^{(1)}(\tau, x) &\equiv \{\phi_0(x), V(\phi^{(0)}(-\tau))\} \\ &= \frac{c^2}{2} \left(\partial_{xx}^2 \phi^{(0)}(-\tau, x) \tau e^{i\tau L(x)/\rho^2} - \partial_{xx}^2 \bar{\phi}^{(0)}(-\tau, x) \frac{\tau}{\rho^2} e^{-i\tau L(x)/\rho^2} \phi_0^2(x) \right) \\ &= \frac{ic^2\tau}{\rho^2} (\rho^2 \partial_{xx}^2 \theta_0(x) - \tau \partial_{xx}^2 L(x)) \phi_0(x),\end{aligned}\quad (5.23)$$

where we have put $\phi_0(x) = \rho e^{i\theta_0(x)}$. Therefore,

$$\phi^{(l)}(t, x) = U_l(t) \phi_0(x) = \phi_0(x) + \int_0^t d\tau \phi_0^{(1)}(\tau, x) + \dots \quad (5.24)$$

$$= \phi_0(x) \left(1 + \frac{ic^2 t^2}{2\rho^2} (\rho^2 \partial_{xx}^2 \theta_0(x) - \frac{2}{3} t \partial_{xx}^2 L(x)) + \dots \right). \quad (5.25)$$

The last step in (5.19), i.e. $\phi(t) = U_0(t) \phi^{(l)}(t)$, is easily performed by replacing $\phi_0(x)$ by $\phi^{(0)}(t, x)$ (and $\theta_0(x)$ by $\theta^{(0)}(t, x) = \theta_0(x) + t L(x)/\rho^2$) everywhere in $\phi^{(l)}(t, x)$. That is:

$$\phi(t, x) = U_0(t) \phi^{(l)}(t, x) = \phi^{(0)}(t, x) \left(1 + \frac{ic^2 t^2}{2} (\partial_{xx}^2 \theta_0(x) + \frac{1}{3\rho^2} t \partial_{xx}^2 L(x)) + \dots \right). \quad (5.26)$$

One can check that, at this order, the perturbative solution coincides with the exact solution $\phi(t, x) = \rho e^{i\theta(t, x)}$ where

$$\theta(t, x) = \cos(ct \partial_x) \theta_0(x) + \frac{\sin(ct \partial_x)}{c \partial_x} \dot{\theta}_0(x). \quad (5.27)$$

Inside the discrete, mechanical picture depicted in Figure 5.1, the appearance of second-order spatial derivatives ∂_{xx}^2 at first order in perturbation theory means that the interaction propagates from one point x_k to its nearest neighbors x_{k+1} and x_{k-1} at this order. In order to account for a longer range propagation we should go to higher orders in perturbation theory.

5.3 Quantum non-canonical perturbation theory

In quantum field theory the fields $\phi(x)$ and $L(x)$ are promoted to the quantum operators $\hat{\phi}$ and \hat{L} , respectively, and the Poisson brackets (5.9) and (5.22) are promoted to the (non canonical) commutators (we shall keep restricting ourselves to $N = 1, D = 2$, for simplicity):

$$[\hat{L}(x), \hat{\phi}(y)] = \hbar \hat{\phi}(x) \delta(x - y), \quad [\hat{L}(x), \hat{\phi}^\dagger(y)] = -\hbar \hat{\phi}^\dagger(x) \delta(x - y), \quad (5.28)$$

where we have introduced \hbar just to account for quantum corrections and $\rho^2 = \hat{\phi}(x) \hat{\phi}^\dagger(x)$ gets the necessary dimensions to render the Hamiltonian with energy dimensions. Let us

consider the lattice picture of our field model and write $\hat{\phi}(x_k) = \hat{\phi}_k$ and $\hat{L}(x_k) = \hat{L}_k$. The Hilbert space $\mathcal{H}_k = \text{Span}(|n_k\rangle, n_k \in \mathbb{Z})$ of a single rotator at position x_k is spanned by the (normalized) eigenstates $|n_k\rangle$ of the angular momentum \hat{L}_k , that is:

$$\hat{L}_k |n_k\rangle = \hbar n_k |n_k\rangle. \quad (5.29)$$

The operators $\hat{\phi}_k$ and $\hat{\phi}_k^\dagger$ act on $|n_k\rangle$ as ladder operators, namely:

$$\hat{\phi}_k |n_k\rangle = \rho |n_k + 1\rangle, \quad \hat{\phi}_k^\dagger |n_k\rangle = \rho |n_k - 1\rangle. \quad (5.30)$$

The total Hilbert space \mathcal{H} of our lattice quantum field theory will be the direct product $\mathcal{H} = \bigotimes_{k \in \mathbb{Z}} \mathcal{H}_k$. The total Hamiltonian operator is

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\omega}{2\hbar} \sum_{k=-\infty}^{\infty} \hat{L}_k^2, \quad \hat{V}(\hat{\phi}) = -\kappa \sum_{k=-\infty}^{\infty} \text{Re}(\hat{\phi}_{k+1} \hat{\phi}_k^\dagger), \quad (5.31)$$

where we have discarded a c-number addend in \hat{V} and we have introduced a frequency $\omega \equiv h\hbar/\rho^2$. In order to write the evolution operator in the interaction image $U_I(t)$, we need to evolve $\hat{\phi}_k$ with the free evolution operator $U_0(t) = e^{-\frac{it}{\hbar} \hat{H}_0}$:

$$\hat{\phi}_k^{(0)}(t) = U_0(-t) \hat{\phi}_k U_0(t) = \sum_{m=0}^{\infty} \frac{(-it/\hbar)^m}{m!} [\hat{\phi}_k, \hat{H}_0]^{(m)}, \quad (5.32)$$

where we denote the multiple commutator:

$$[\hat{\phi}_k, \hat{H}_0]^{(m)} \equiv [[\hat{\phi}_k, \hat{H}_0], \dots, \hat{H}_0]. \quad (5.33)$$

The quantum commutator introduces new ordering problems with respect to the classical Poisson bracket. For standard creation \hat{a}_k^\dagger and annihilation \hat{a}_k operators, Wick's theorem provides a useful tool for writing arbitrary products of \hat{a}_k^\dagger and \hat{a}_l in terms of normal ordered products. Here we have to deduce a new Wick-like theorem in order to write arbitrary products of the non-canonical operators \hat{L}_k and $\hat{\phi}_l$. If we choose by convention to write all \hat{L} 's to the left of all $\hat{\phi}$'s, then the multiple commutator (5.33) acquires the following form:

$$[\hat{\phi}_k, \hat{H}_0]^{(m)} = \frac{(-1)^m \omega^m}{2^m} \left(\sum_{l=0}^m c_{m,l} \hbar^l \hat{L}_k^{m-l} \right) \hat{\phi}_k = (-1)^m \omega^m (\hat{L}_k^m + q.c.) \hat{\phi}_k, \quad (5.34)$$

where *q.c.* stands for "quantum corrections". The Wick-like numerical coefficients $c_{m,l}$ are given by $c_{m,0} = 2^m$, $c_{m,m} = (-1)^m$ and the recurrence $c_{m,l} = 2c_{m-1,l} - c_{m-1,l-1}$. Therefore

$$\hat{\phi}_k^{(0)}(t) = U_0(-t) \hat{\phi}_k U_0(t) = (e^{it\omega \hat{L}_k/\hbar} + q.c.) \hat{\phi}_k, \quad (5.35)$$

coincides with the classical expression (5.21) except for quantum corrections. Actually, we shall be able to sum up all quantum corrections in some particular cases (see later) by noticing that

$$\sum_{l=0}^m c_{m,l} = 1, \quad \forall m = 0, 1, 2, \dots \quad (5.36)$$

The evolution operator in the interaction image (5.18) is given in terms of

$$\hat{V}(\hat{\phi}^{(0)}(-\tau)) = -\kappa \sum_{q=-\infty}^{\infty} \text{Re} \left((e^{-i\tau\omega\hat{L}_{q+1}/\hbar} + q.c.) \hat{\phi}_{q+1} \hat{\phi}_q^\dagger (e^{i\tau\omega\hat{L}_q/\hbar} + q.c.) \right). \quad (5.37)$$

In order to describe the new perturbation scheme, let us consider an initial state (at time $t = 0$)

$$|\{n\}\rangle = \otimes_{q \in \mathbb{Z}} |n_q\rangle. \quad (5.38)$$

The probability amplitude of observing $|\{n'\}\rangle$ as a final state after time t is given by the S matrix element:

$$S_{n,n'}(t) = \langle \{n'\} | U(t) | \{n\} \rangle = \langle \{n'\} | U_0(t) U_I(t) | \{n\} \rangle = e^{it\frac{\omega}{2} \sum_{k=-\infty}^{\infty} (n'_k)^2} \langle \{n'\} | U_I(t) | \{n\} \rangle. \quad (5.39)$$

The total angular momentum $\hat{L} = \sum_k \hat{L}_k$ is conserved at all orders in perturbation theory since $[\hat{L}, \hat{V}] = 0$. This means that

$$\sum_{k=-\infty}^{\infty} n_k \neq \sum_{k=-\infty}^{\infty} n'_k \Rightarrow S_{n,n'}(t) = 0. \quad (5.40)$$

The interaction potential (5.37) is of short range, that is, \hat{V} is not able to carry one quantum of angular momentum from position k to l until $|k - l|$ -th order in perturbation theory. More precisely, considering an initial state of the form

$$|\{\delta_k\}\rangle = \otimes_{q \in \mathbb{Z}} |\delta_{k,q}\rangle, \quad (5.41)$$

we can compute the probability amplitude of observing $|\{\delta_l\}\rangle$ as a final state after time t at all orders:

$$S_{\delta_k, \delta_l}(t) = e^{it\omega/2} \sum_{n=0}^{\infty} \left(\frac{it\kappa\rho^2}{2\hbar} \right)^n \sum_{s=0}^n \binom{n}{s} \delta_{l, k-n+2s}, \quad (5.42)$$

where we have made use of (5.36) at some stage. Note that perturbation theory is dictated by both: κ and/or ρ . Instead of the angular momentum eigenstates $|n_k\rangle$ we could also have used field eigenstates

$$|\zeta_k\rangle \equiv \sum_{n=-\infty}^{\infty} \zeta_k^n |n\rangle, \quad |\zeta_k| = 1, \quad (5.43)$$

for which $\hat{\phi}_k |\zeta_k\rangle = \rho \zeta_k |\zeta_k\rangle$, $\hat{\phi}_k^\dagger |\zeta_k\rangle = \rho \zeta_k^{-1} |\zeta_k\rangle$ and $\hat{L}_k |\zeta_k\rangle = \hbar \zeta_k \partial_{\zeta_k} |\zeta_k\rangle$. Moreover, going from $N = 1$ to arbitrary N can be accomplished by replacing $|n_k\rangle$ with hyper-spherical harmonics. For $N = 2$, the usual spherical harmonics are given in terms of homogeneous polynomials of degree j in $\vec{\phi}$:

$$Y_m^j(\vec{\phi}_k) = \sum_{\substack{a_q = 1, 2, 3 \\ q = 1, \dots, j}} \xi_{a_1, \dots, a_j}^{(m)} \phi_k^{a_1} \dots \phi_k^{a_j}, \quad (5.44)$$

where $\xi_{a_1, \dots, a_j}^{(m)}$ are the complex components of a symmetric and traceless tensor [112]. The angular momentum operator at place x_k is then given by $\hat{L}_k^a = \hbar \epsilon^{abc} \phi_k^c \partial_{\phi_k^b}$, as usual.

The Quantum Arnold Transformation and Dissipation

The interest in dissipative systems at the quantum level has remained constant since the early days of Quantum Mechanics. The difficulties in describing damping, which intuitively could be understood as a mesoscopic property, within the fundamental quantum framework, have motivated a huge amount of papers.

Applications of quantum dissipation abound. For example, in quantum optics, where the quantum theory of lasers and masers makes use of models including damping [153], or in the study of decoherence phenomena [154]. Some authors have modeled dissipation by means of the theory of open systems or the thermal bath approach, in which a damped system is considered to be a subsystem of a bigger one with infinite degrees of freedom [155, 154]. However, damped systems are interesting in themselves as fundamental ones. In particular, the quantum damped harmonic oscillator, frequently described by the Caldirola-Kanai equation [113, 114], has attracted much attention, as it could be considered one of the simplest and paradigmatic examples of dissipative system.

The description of the quantum damped harmonic oscillator by the Caldirola-Kanai model, which includes a time-dependent Hamiltonian, has been considered to have some flaws. For instance, it is claimed that uncertainty relations are not preserved under time evolution and could eventually be violated [119, 120]. Many considerations were made in this direction. For example, Dekker in [156] introduced complex variables and noise operators to tackle the problem, claiming that no dynamical description in terms of a Schrödinger wave function can be expected to exist. In [157], a non-linear Schrödinger-Langevin wave equation was proposed as the starting point in formulating the quantum theory. However, this inconsistency seems to be associated with a confusion between canonical momentum and “physical” momentum [121].

Despite these considerations about the Caldirola-Kanai model, many developments went ahead. Coherent states were calculated in [118] by finding creation and annihilation operators, built out of operators which commute with the Schrödinger equation. The corresponding number operator turns out to be an auxiliary, conserved operator, obviously different from the time-dependent Hamiltonian. This paper also defined the so-called loss energy states for the damped harmonic oscillator. The famous report by Dekker [158] provides a historical overview of some relevant results.

The analysis of damping from the symmetry point of view has proved to be especially

fruitful. In a purely classical context, the symmetries of the equation of the damped harmonic oscillator with time-dependent parameters were found in [115]. Two comprehensive articles, [116, 132], are of special interest. In those papers the authors found, for the damped harmonic oscillator, finite-dimensional point symmetry groups for the corresponding Lagrangian (the un-extended Schrödinger group [159]) and the equations of motion ($SL(3, \mathbb{R})$) respectively, and an infinite contact one for the set of trajectories of the classical equation. They singled out a “non-conventional” Hamiltonian from those generators of the symmetry, recovering some results from [118]. Then, they concluded that the damped harmonic oscillator should not be claimed to be dissipative at all at the quantum level, as this true, “non-conventional” Hamiltonian is conserved, and should be related to an oscillator with variable frequency.

Many papers related to the Caldirola-Kanai model keep appearing, showing that the debate about fundamental quantum damping is far from being closed. We can mention [161], where the driven damped harmonic oscillator is analyzed, or the review [162]. Even the possible choices of classical Poisson structures and Hamiltonians, or generalizations to the non-commutative plane, have deserved attention as recently as in [163] and [164], respectively.

There exists another interesting approach to the study of the classical damped harmonic oscillator, based on the observation that its classical equation of motion is a special case of the set of linear second-order ordinary differential equations (LSODE for short). In Classical Mechanics the family of solutions of a second-order differential equation corresponding to the motion of a given physical problem is sometimes related to that of a simpler system, considered as a toy model, in order to import from it simple general properties which could be hidden in the real problem. Both physical systems should share global properties of the solution manifold, such as topology and symplectic structure. The paradigmatic example is the transformation described by Arnold in [122], which brings any LSODE to the simplest form of the free Galilean particle equation. This transformation turns out to be extremely useful. In particular, it is possible to obtain the symmetry group of a particular instance of LSODE [115], in which the symmetries of the action of the corresponding system can be found as a subgroup [123].

Therefore, it seems natural to try to generalize the Arnold transformation to the quantum level, to be denoted as Quantum Arnold Transformation (QAT), as much insight can be gained in the study of any system classically described by a LSODE and, in particular, the parametric oscillator or some of the systems which present dissipation.

Several partial generalizations can be found in the literature. For example, in [124], Takagi is able to provide a transformation which relates the Schrödinger equation of the harmonic oscillator to that of the free particle, and applies it to simplify the computation of the propagator by making use of the free one. [125] contains a slightly more general version (see formula (33) therein). [126] went a bit further considering the damped harmonic oscillator with constant parameters. In [127] the particle with time-dependent mass in a linear potential was studied by unitarily relating it to the free particle, and this was used to define Airy-like wave packets for the free particle. None of them mentions the classical Arnold transformation, but it *is* underlying their reasoning.

Implicitly, a generalization of the Arnold transformation was also contained in [118], the classical version not being referred once more. It will be shown that some of their

results formally converge with ours (see Section 6.1.3.2), although they put emphasis on another aspects of the problem, such as the analysis of unitarity and energy loss. Mostafazadeh [128] also pursued the idea of “connecting” different quantum physical systems by means of time-dependent unitary transformations, even representing arbitrary time-dependent diffeomorphisms [129]. His approach is rather general, but does not fully take advantage of the possibility of connecting with the free particle system and importing its symmetries.

Besides the Caldirola-Kanai model, the Bateman’s dual system appears as an alternative description of dissipation in the damped harmonic oscillator. In his original paper [165], Bateman looked for a variational principle for equations of motion with a friction term linear in velocity, but he allowed the presence of extra equations. This trick effectively doubles the number of degrees of freedom, introducing a time-reversed version of the original damped harmonic oscillator, which acts as an energy reservoir and could be considered as an effective description of a thermal bath. The Hamiltonian that describes this system was rediscovered by Feschbach and Tikochinsky [166, 167, 158] and the corresponding quantum theory was immediately analyzed.

Some issues regarding the Bateman’s system arose. The Hamiltonian presents a set of complex eigenvalues of the energy (see [169] and references therein), and the vacuum of the theory decays with time. This last feature was treated in [168], where Celeghini et al. suggested that the quantum theory of the dual system could find a more natural framework in quantum field theory¹. On the other hand, in [169] the generalized eigenvectors corresponding to the complex eigenvalues are interpreted as resonant states.

Bateman’s dual system is still frequently discussed [170]. Many authors have considered this model as a *good* starting point for the formulation of the quantum theory of dissipation. One of the aims of this chapter will be to show that the study of the symmetries of the Caldirola-Kanai model leads to the Bateman’s dual system, thus to be considered as a *natural* starting point for the study of quantum dissipation.

6.1 The Arnold transformation

6.1.1 Classical Arnold transformation

Mathematically speaking, the classical Arnold transformation [122] converts any linear second-order ordinary differential equation (LSODE) into the free Galilean particle equation, that is, $\ddot{\kappa} = 0$ in 1+1 dimensions (we shall limit ourselves to this situation).

From the physical point of view, the Arnold transformation relates the trajectories $x(t)$, with initial conditions x_0 and $p_0 \sim \dot{x}_0$, solutions of the LSODE, to those trajectories $\kappa(\tau)$ solutions of the free equation with initial conditions κ_0 and $\pi_0 \sim \dot{\kappa}_0$. Either (x_0, p_0) or (κ_0, π_0) parametrize the common solution manifold \mathcal{M} , and we shall adopt the unified notation (K, P) . On this manifold, each physical system is characterized by the corresponding Hamiltonian as a function of K and P . The inverse of the corresponding

¹We feel that the ultimate reason is nevertheless the lack of a vacuum representation of the relevant group (see Section 6.3).

Hamilton-Jacobi transformation then recovers the trajectories $(x(t), \dot{x}(t))$ or $(\kappa(\tau), \dot{\kappa}(\tau))$ out of the K, P variables.

Following a similar notation to that in [115], we give an overview of the Arnold transformation [122]. Firstly, let us recall that, given an arbitrary, non-homogeneous LODE

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda, \quad (6.1)$$

where $\dot{x} = \frac{dx}{dt}$ and so on, and f, ω and Λ are arbitrary functions of time t , we can apply the transformation

$$\begin{cases} t \longrightarrow t \\ x \longrightarrow x + u_p \end{cases}, \quad (6.2)$$

u_p being a particular solution of (6.1). We find that the differential equation above is transformed into

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = 0, \quad (6.3)$$

i. e., every non-homogeneous problem is equivalent to a homogeneous one.

The homogeneous Arnold transformation, is a local diffeomorphism which maps the free particle equation of motion into (6.3):

$$\begin{cases} \tau = \frac{u_1(t)}{u_2(t)} \\ \kappa = \frac{x}{u_2(t)} \end{cases}, \quad \ddot{\kappa} = 0 \quad \longleftrightarrow \quad \ddot{x} + \dot{f}\dot{x} + \omega^2 x = 0, \quad (6.4)$$

where $u_1(t)$ and $u_2(t)$ are independent solutions of (6.3). Applying the inverse diffeomorphism to the classical dynamical system (6.3), we can transform this equation into the free one.

If we include external forces the transformation (6.4) turns into the general Arnold transformation, that we shall call simply A :

$$\begin{cases} \tau = \frac{u_1(t)}{u_2(t)} \\ \kappa = \frac{x - u_p(t)}{u_2(t)} \end{cases}, \quad \ddot{\kappa} = 0 \quad \longleftrightarrow \quad \ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda. \quad (6.5)$$

This transformation could be understood as passing to coordinates analogous to co-moving spacial coordinate and proper time used in General Relativity, so that the system becomes “free”, at least locally.

Indeed this transformation is of *local* nature in time, in the sense that it is only valid for an open interval in time t . In fact, it can be shown that the equivalence in (6.4) and (6.5) is true up to a factor $\frac{u_2^2}{e^{-2f}}$, so that it holds in the interval where u_2 does not vanish. This means that the transformation does not take the Euler-Lagrange operator associated with the LODE itself to that of the free system. For this reason it can not be claimed that both physical systems are actually equivalent.

However, Arnold transformation can help to understand the physical system under study. In particular, as pointed in [115], it is possible to identify the set of contact symmetries for (6.3), and this way to arrive at the results found in [116], which show the sets

of symmetries for either the equations or the action from which such equations can be derived.

It should be noted that, due to the general character of the transformation, we could miss the physical identity of position and time when performing such a transformation. But it will be possible to choose appropriate specific solutions u_1 , u_2 and u_p with suitable initial conditions so that the identity of the variables is maintained². We shall make use of this possibility in going to the quantum version of this transformation.

Finally, having in mind the particular case of dissipative systems, we would like to remark that certain issues of the treatment of these systems are already apparent in the classical domain. For instance, time evolution is not a symplectomorphism, nor preserve the volume of the phase space. Obviously, the Hamiltonian function is not a Noether invariant. This becomes especially manifest and annoying when formulating the quantum theory.

6.1.2 The Quantum Arnold transformation

As already mentioned in the Introduction, several partial versions of the QAT can be found in the literature. Here we give a generalization that contains, as particular cases, those found in [124, 125, 126, 127, 118].

In bringing Arnold's technique to the quantum world we must be aware, obviously, of the different philosophy of the quantum description and different nature of the equation of motion. The objects and structures that define a quantum system, namely the Hilbert space, the basic observables, the Hamiltonian operator, and the Schrödinger equation must be specified in a way that we are able to identify the same objects at both sides of the transformation.

To this end, it is important to focus, in the free system, only on those operators corresponding to constants of motion, Noether invariants associated with its symmetry, that is, the Schrödinger group (which contains the centrally-extended Galilei group as a subgroup, containing in turn the Heisenberg-Weyl group of translations and non-relativistic boosts). This implies to fix the basic operators (that is to say, quantum operators which realize a unitary and irreducible representation of the common classical Poisson (Heisenberg-Weyl) algebra $\{K, P\} = 1$) so that they respect the Schrödinger equation, then having constant expectation values and being generators of the basic symmetry³ (the Heisenberg-Weyl sub-group of the centrally-extended Schrödinger group).

Those basic operators, in this form, are in principle the candidates to be related by a quantum version of the Arnold transformation, so that we shall have the situation as follows: On the one hand, a common Hilbert space \mathcal{H} of wave functions $\Psi(K)$ ($L^2(\mathbb{R})$), which plays the role of initial values for both the solutions $\phi(x, t) \in \mathcal{H}_t$ of the Schrödinger equation relative to the quantum version of our original LSODE and those wave functions, $\varphi(\kappa, \tau) \in \mathcal{H}_\tau^G$, solutions of the free Schrödinger equation. And, on the other hand, the

²For his restricted version of the quantum Arnold transformation, Takagi, in [124], suggested that the transformation could be specified by different solutions at different times, in principle avoiding the restriction of locality. However, care must be taken when making use of this freedom to prevent conflict with preserving the identity of the variables when it is desirable.

³By "basic symmetry" we understand, in general, those symmetries whose associated Noether invariants are enough to parametrize the classical solution manifold.

quantum Arnold transformation \hat{A} relating Schrödinger equations and basic operators. As a crucial consequence, we shall obtain a realization of the free symmetry on the quantum, non-free system. The following diagram can help to have a picture of the setup:

$$\begin{array}{ccc}
 \mathcal{H}_\tau^G & \xleftarrow{\hat{A}} & \mathcal{H}_t \\
 \tilde{U}_G(\tau) \uparrow & & \uparrow \tilde{U}(t) \\
 \mathcal{H}_0^G \equiv \mathcal{H} & \xrightarrow{\hat{1}} & \mathcal{H} \equiv \mathcal{H}_0
 \end{array} \tag{6.6}$$

\mathcal{H}_τ^G (resp. \mathcal{H}_t) is the Hilbert space of solutions of the free or Galilean (resp. non-free or corresponding to the LSODE) Schrödinger equation, $\tilde{U}_G(\tau)$ (resp. $\tilde{U}(t)$) is the free (resp. non-free) evolution operator and $\hat{1}$ is the identity operator. Here, the Hilbert space \mathcal{H} may be considered as the quantum analogue of the classical solution manifold, \mathcal{M} , usually thought of as space of (classical) initial conditions. On \mathcal{H} one must be able to measure all possible physical observables in much the same way classical observables are characterized as real functions on \mathcal{M} , that is, functions whose arguments are constants along classical trajectories (functions of Noether invariants).

The Hamiltonian of the non-free system, not being conserved in general, will not be related to any operator from the free particle. This is to be expected, since it is not a conserved quantity under the evolution of the physical system, neither at classical nor at the quantum level (in the sense that it does not have constant expectation values). It is important to remark that this implies that it is not possible to formulate a time-independent Schrödinger equation.

More specifically, by extending properly the Arnold transformation (or the inverse) to the quantum case, we shall relate the space \mathcal{H}_τ^G of solutions of the free Schrödinger equation

$$i\hbar \frac{\partial \varphi}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial \kappa^2}, \tag{6.7}$$

with corresponding classical equation

$$\ddot{\kappa} = 0, \tag{6.8}$$

to that space \mathcal{H}_t where the quantum theory of the generic LSODE

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda, \tag{6.9}$$

is realized, the quantities f , ω and Λ being, in general, time-dependent.

The classical equation (6.9) can be derived from a variational principle. We shall consider the Lagrangian

$$L = e^f \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + m \Lambda x \right) \tag{6.10}$$

as our starting point. The Schrödinger equation can be derived from the corresponding classical Hamiltonian function

$$H = \frac{p^2}{2m} e^{-f} + \left(\frac{1}{2} m \omega^2 x^2 - m \Lambda x \right) e^f, \tag{6.11}$$

according to the standard canonical prescriptions, leading to

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} e^{-f} \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{1}{2} m \omega^2 x^2 - m \Lambda x\right) e^f \phi. \quad (6.12)$$

For f linear in time, constant ω and $\Lambda = 0$ this equation is commonly known as Caldirola-Kanai equation for the damped harmonic oscillator [113, 114].

Even though both spaces of solutions of (6.7) and (6.12), \mathcal{H}_τ^G and \mathcal{H}_t respectively, will be related, and the basic quantum operators associated with the classical functions K, P realized as well-defined operators both on \mathcal{H}_τ^G and \mathcal{H}_t , we cannot still assure that both physical systems are actually equivalent. In fact, the evolution operator in time t , $i\hbar \frac{\partial}{\partial t}$, does definitely not leave invariant the space of the solutions of (6.12) in general, nor comes down to the space \mathcal{H} , which means that it cannot be realized as an operator function of \hat{K}, \hat{P} (in sharp contrast to $i\hbar \frac{\partial}{\partial \tau}$, which is $\sim \hat{P}^2$). We shall achieve the construction of well-defined Hermitian evolution-like generators imported from the free system via the inverse quantum Arnold transformation, but their eigenvalues, conserved indeed, do not correspond to the standard energy⁴. These operators close the Schrödinger algebra with the basic operators.

Implicitly, this trick of considering operators different from the Hamiltonian to provide quantum numbers and obtain solutions of the Schrödinger equation as their eigenfunctions has been used extensively. For example, the operator found in [118] to label the energy-loss states, which coincides with the quantum operator H^* corresponding to G_5 in [116, 132], turns out to be a generator of the $SL(2, \mathbb{R})$ subgroup of the Schrödinger group. We give here explicitly the frame in which this can be done: operators from the Schrödinger group can be chosen to play this role upon convenience.

For the sake of simplicity, let us focus on the case with no external forces $\Lambda = 0$. The formulas corresponding to $\Lambda \neq 0$ are given in Subsection 6.1.4.

The generalization of the classical Arnold transformation is obtained by completing (6.4) with a change of the wave function. Explicitly, the quantum Arnold transformation, valid for every physical system with classical equation of the homogeneous LSODE type, is given by the (local) diffeomorphism:

$$\begin{cases} \tau = \frac{u_1(t)}{u_2(t)} \\ \kappa = \frac{x}{u_2(t)} \\ \varphi = \phi \sqrt{u_2(t)} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{W(t)} \frac{\dot{u}_2(t)}{u_2(t)} x^2}, \end{cases} \quad (6.13)$$

where u_1 and u_2 satisfy again the classical equation of motion in (x, t) , $\dot{u}_1 = \frac{du_1}{dt}$, $\dot{u}_2 = \frac{du_2}{dt}$ and $W(t) \equiv \dot{u}_1 u_2 - u_1 \dot{u}_2 = e^{-f}$. It is straightforward to check that by this transformation the Schrödinger equation of the free particle is transformed into (6.12) (with $\Lambda = 0$) up to a multiplicative factor which depends on the particular choice of the classical solutions u_1 and u_2 (partial derivatives must be changed by the classical part of the transformation while wave functions are shifted by the quantum part).

⁴The construction of a properly defined, Hermitian generator in standard time will take much effort requiring a sound analysis of the symmetry problem in damped systems. This has been done in [131].

Now, we can impose on u_1 and u_2 the condition that they preserve the identity of τ and κ , i.e., that (κ, τ) coincide with (x, t) at an initial point t_0 , arbitrarily taken to be $t_0 = 0$:

$$u_1(0) = 0, \quad u_2(0) = 1, \quad \dot{u}_1(0) = 1, \quad \dot{u}_2(0) = 0. \quad (6.14)$$

This fixes a unique form of the diffeomorphism for a given ‘‘target’’ physical system. However, the quantum Arnold transformation would still be valid if solutions u_1 and u_2 do not satisfy (6.14). The price to be paid would then be that the relation in the lower part of the diagram (6.6) above would no longer be the identity and basic position and momentum operators would then be mixed (see the end of this Section).

Formally, while the classical Arnold transformation in this case is:

$$A: \mathbb{R} \times T \longrightarrow \mathbb{R} \times T'$$

$$(x, t) \longmapsto (\kappa, \tau) = A((x, t)) = \left(\frac{x}{u_2}, \frac{u_1}{u_2} \right), \quad (6.15)$$

where T and T' are open intervals of the real line containing $t = 0$ and $\tau = 0$, respectively, QAT can be written:

$$\hat{A}: \mathcal{H}_t \longrightarrow \mathcal{H}_\tau^G$$

$$\phi(x, t) \longmapsto \varphi(\kappa, \tau) = \hat{A}(\phi(x, t)) = A^* \left(\sqrt{u_2(t)} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{w(t)} \frac{\dot{u}_2(t)}{u_2(t)} x^2} \phi(x, t) \right), \quad (6.16)$$

where A^* denotes the pullback operation corresponding to A .

As already remarked, the basic symmetries of the free system are inherited by the LSODE-type system, as we are now able to transform the infinitesimal generators of translations (the Galilean momentum operator $\hat{\pi}$, corresponding to the classical conserved quantity ‘momentum’) and non-relativistic boosts (the position operator $\hat{\kappa}$, corresponding to the classical conserved quantity ‘initial position’). They are, explicitly,

$$\hat{\pi} = -i\hbar \frac{\partial}{\partial \kappa} \quad (6.17)$$

$$\hat{\kappa} = \kappa + \frac{i\hbar}{m} \tau \frac{\partial}{\partial \kappa}, \quad (6.18)$$

that is, those basic, canonically commuting operators with constant expectation values, that respect the solutions of the free Schrödinger equation, have constant matrix elements (and constant expectation values in particular) and fall down to well defined, time-independent operators in the Hilbert space of the free particle $L^2(\mathbb{R})$.

In general, these properties are satisfied whenever an operator $\hat{O}(t)$ can be written as

$$\hat{O}(t) = \hat{U}(t, t_0) \hat{O} \hat{U}^\dagger(t, t_0), \quad (6.19)$$

where \hat{O} is $\hat{O}(t_0)$ and $\hat{U}(t, t_0)$ is the evolution operator satisfying the Schrödinger equation⁵. If the Hamiltonian is time-independent, as in the free particle case, time-evolution is a one-parameter group and then $\hat{U}(\tau, \tau_0) = \hat{U}(\tau - \tau_0)$.

⁵Note that $\hat{O}(t)$ is *not* the usual Heisenberg picture version $\hat{O}_H(t) = \hat{U}(t, t_0)^\dagger \hat{O} \hat{U}(t, t_0)$ of its associated operator in Schrödinger picture \hat{O} , although their relation is very simple when the Hamiltonian is time-independent.

Defining a generic Schrödinger equation operator, $\hat{S} \equiv i\hbar \frac{\partial}{\partial t} - \hat{H}$, taking $t_0 = 0$ ($\hat{U}(t) \equiv \hat{U}(t, 0)$) and reminding that $\hat{U}(t, t_0)^\dagger \hat{U}(t, t_0) = 1$, it is clear that, for operators of the form (6.19):

$$\hat{S} \hat{O}(t) |\psi(t)\rangle = \hat{S} \hat{U}(t) \hat{O} |\psi\rangle \equiv \hat{S} \hat{U}(t) |\psi'\rangle = \hat{S} |\psi'(t)\rangle = 0 \quad (6.20)$$

$$\frac{\partial}{\partial t} \langle \chi(t) | \hat{O}(t) | \psi(t) \rangle = 0 \quad (6.21)$$

$$\frac{\partial}{\partial t} (\hat{U}(t)^\dagger \hat{O}(t) \hat{U}(t)) = 0 \quad (6.22)$$

$$\frac{d}{dt} \hat{O}(t) \equiv \frac{\partial}{\partial t} \hat{O}(t) + \frac{i}{\hbar} [\hat{H}(t), \hat{O}(t)] = 0 \quad (6.23)$$

(where $|\psi(t)\rangle \equiv \hat{U}(t) |\psi\rangle$), stating that those operators respect solutions, have constant matrix elements, fall down to define time-independent operators on the Hilbert space and are integrals of the motion, respectively. (6.17) and (6.18) can be “de-evolved”, so that

$$\hat{U}(\tau)^\dagger \hat{\pi} \hat{U}(\tau) = -i\hbar \frac{\partial}{\partial \kappa}, \quad \hat{U}(\tau)^\dagger \hat{\kappa} \hat{U}(\tau) = \kappa, \quad (6.24)$$

thus showing the properties above for the free particle.

What we are doing is to focus on these integrals of motion, $\hat{\pi}$ and $\hat{\kappa}$, so that the new operators position \hat{X} and momentum \hat{P} acting on \mathcal{H}_t are also integrals of motion in the non-free system. These will be the generators of the basic symmetry in the non-free system. Dodonov and Man'ko [118] obtained these operators in particular cases by direct calculation, imposing them to commute with the Schrödinger equation \hat{S} . The difference is that, having related this system with that of the free particle, now it is clear how far one can go: the Schrödinger group and its enveloping algebra. Even more, the approach followed in [118] is only able to provide the basic operators (corresponding to linear functions in the classical solution manifold), since the condition of commuting with \hat{S} , $[\hat{S}, \hat{O}(t)] = 0$, is more restrictive than that considered here of respecting solutions, which is equivalent to $[\hat{S}, \hat{O}(t)] \sim \hat{S}$.

Let us apply the QAT (6.13) to (6.17) and (6.18). For a given operator $\hat{\pi}$ acting on \mathcal{H}_τ^G there is a corresponding operator $\hat{P} = \hat{A}^{-1} \hat{\pi} \hat{A}$ on \mathcal{H}_t . The action on functions $\phi(x, t)$ can then be obtained as follows:

$$\begin{aligned} \hat{P} \phi(x, t) &= \hat{A}^{-1} \hat{\pi} \hat{A} \phi(x, t) = \hat{A}^{-1} \hat{\pi} A^* (\sqrt{u_2} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} \phi(x, t)) = \\ &= \frac{1}{\sqrt{u_2}} e^{+\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} A^{*-1} \left(-i\hbar \frac{\partial}{\partial \kappa} A^* (\sqrt{u_2} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} \phi(x, t)) \right) = \\ &= \frac{1}{\sqrt{u_2}} e^{+\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} \left(-i\hbar u_2 \frac{\partial}{\partial x} (\sqrt{u_2} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} \phi(x, t)) \right) = \\ &= (-i\hbar u_2 \frac{\partial}{\partial x} - m \frac{\dot{u}_2}{W} x) \phi(x, t). \end{aligned} \quad (6.25)$$

We can perform the same computation for the position operator and then we have:

$$\hat{P} = -i\hbar u_2 \frac{\partial}{\partial x} - m x \frac{\dot{u}_2}{W} \quad (6.26)$$

$$\hat{X} = \frac{\dot{u}_1}{W} x + \frac{i\hbar}{m} u_1 \frac{\partial}{\partial x}, \quad (6.27)$$

thus providing the generators of the realization of the (centrally-extended) Heisenberg-Weyl symmetry on the physical system corresponding to a general LODE.

The properties of the operators (6.26) and (6.27), i.e. preserving solutions of (6.12), having constant matrix elements, falling to the Hilbert space, are ensured by the properties of (6.17) and (6.18) before the transformation, and will be explicitly checked for some particular cases in next Section. It will also become clear that the identity of both operators is preserved after the transformation.

Apart from \hat{P} and \hat{X} , we can compute \hat{P}^2 , \hat{X}^2 and $\hat{X}\hat{P} \equiv \frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})$:

$$\hat{P}^2 = -\hbar^2 u_2^2 \frac{\partial^2}{\partial x^2} + i\hbar \frac{2m u_2 \dot{u}_2}{W} x \frac{\partial}{\partial x} + m^2 \frac{\dot{u}_2^2}{W^2} x^2 + i\hbar \frac{m u_2 \dot{u}_2}{W} \quad (6.28)$$

$$\hat{X}^2 = \frac{\dot{u}_1^2}{W^2} x^2 + i\hbar \frac{2u_1 \dot{u}_1}{mW} x \frac{\partial}{\partial x} - \hbar^2 \frac{u_1^2}{m^2} \frac{\partial^2}{\partial x^2} + i\hbar \frac{u_1 \dot{u}_1}{mW} \quad (6.29)$$

$$\hat{X}\hat{P} = \frac{\hbar^2}{m} u_1 u_2 \frac{\partial^2}{\partial x^2} - i\hbar \frac{\dot{u}_1 u_2 + u_1 \dot{u}_2}{W} x \frac{\partial}{\partial x} - m \frac{\dot{u}_1 \dot{u}_2}{W^2} x^2 - i\hbar \frac{\dot{u}_1 u_2 + u_1 \dot{u}_2}{2W}. \quad (6.30)$$

Their first-order versions, valid on solutions of (6.12), are:

$$\hat{P}^2 = i\hbar \frac{2m u_2^2}{W} \frac{\partial}{\partial t} + i\hbar \frac{2m u_2 \dot{u}_2}{W} x \frac{\partial}{\partial x} + m^2 \frac{\dot{u}_2^2 - \omega^2 u_2^2}{W^2} x^2 + i\hbar \frac{m u_2 \dot{u}_2}{W} \quad (6.31)$$

$$\hat{X}^2 = \frac{\dot{u}_1^2 - \omega^2 u_1^2}{W^2} x^2 + i\hbar \frac{2u_1 \dot{u}_1}{mW} x \frac{\partial}{\partial x} + i\hbar \frac{2u_1^2}{mW} \frac{\partial}{\partial t} + i\hbar \frac{u_1 \dot{u}_1}{mW} \quad (6.32)$$

$$\hat{X}\hat{P} = -i\hbar \frac{2u_1 u_2}{W} \frac{\partial}{\partial t} - i\hbar \frac{\dot{u}_1 u_2 + u_1 \dot{u}_2}{W} x \frac{\partial}{\partial x} - m \frac{\dot{u}_1 \dot{u}_2 - \omega^2 u_1 u_2}{W^2} x^2 - i\hbar \frac{\dot{u}_1 u_2 + u_1 \dot{u}_2}{2W}, \quad (6.33)$$

which, together with \hat{X} and \hat{P} , close the whole Schrödinger Lie algebra:

$$[\hat{X}, \hat{P}] = i\hbar \quad (6.34)$$

$$[\hat{X}, \hat{P}^2] = 2i\hbar \hat{P} \quad [\hat{X}, \hat{X}^2] = 0 \quad [\hat{X}, \hat{X}\hat{P}] = i\hbar \hat{X} \quad (6.35)$$

$$[\hat{P}, \hat{P}^2] = 0 \quad [\hat{P}, \hat{X}^2] = -2i\hbar \hat{X} \quad [\hat{P}, \hat{X}\hat{P}] = -i\hbar \hat{P} \quad (6.36)$$

$$[\hat{X}^2, \hat{P}^2] = 4i\hbar \hat{X}\hat{P} \quad [\hat{X}^2, \hat{X}\hat{P}] = 2i\hbar \hat{X}^2 \quad [\hat{P}^2, \hat{X}\hat{P}] = -2i\hbar \hat{P}^2. \quad (6.37)$$

All these operators are well-defined on the solution space of the time-dependent Schrödinger equation, so that the action of one of them on a solution is again a solution. However, it is important to note once again that the Hamiltonian operator corresponding to the LODE, that is, the quantum version of (6.11) ($\Lambda = 0$), \hat{H} , although being a second-order differential operator, can not be expressed in terms of these operators in general and then it does not close a Lie algebra with them. But that which is worse, it is not even a well-defined operator on the space of solutions of the Schrödinger equation, \mathcal{H}_t . As a consequence, \hat{H} is not the generator of a one parameter group corresponding to conventional time evolution.

Instead, any linear combination of \hat{P}^2 , \hat{X}^2 and $\hat{X}\hat{P}$, say \hat{H} , can be adopted as a well-defined, Hermitian evolution-like generator. It has an associated eigenvalue equation and real spectrum, and its eigenvalues can be used to label solutions of (6.12) ($\Lambda = 0$) as its eigenfunctions. The particular choice of \hat{H} to be taken depends purely on convenience and, for example, the similarity with the form of \hat{H} of the particular physical system.

We would like to point out that there is an essential difference between the approach followed in [132] and ours. The reason is that the un-extended Schrödinger group is considered there as the fundamental symmetry of the damped harmonic oscillator, the origin of which is the analysis of the classical equations of motion in [116]. The approach based on QAT provides directly a representation of a central extension of the Schrödinger group adapted to the specific LODE-type system⁶. For the relevance of central extensions in Quantum Mechanics, we refer to [130, 20, 133, 7, 134, 34].

Let us stress that QAT can be useful to quickly perform some calculations, avoiding tedious, direct evaluations which can become extremely involved in the system under study. For example, it can be used to compute the quantum propagator for any LODE-type quantum system, following the idea of Takagi in [124] for the simple case of the harmonic oscillator, or even the evolution operator $\hat{U}(t)$, which becomes very difficult to evaluate exactly when the Hamiltonian is time-dependent and does not commute with itself at different times.

Actually, the evolution operator of a LODE system can be related with the free evolution operator. Having in mind the diagram (6.6), we write:

$$\hat{A}(\hat{U}(t)\phi(x)) = \hat{U}_G(\tau)\varphi(\kappa). \quad (6.38)$$

Here ϕ and φ are *the same* function of only one argument (κ or x) and we will denote $\varphi = \phi = \psi$. Then,

$$\begin{aligned} \hat{U}(t)\psi(x) &= \hat{A}^{-1}(\hat{U}_G(\tau)\psi(\kappa)) = \frac{1}{\sqrt{u_2}} e^{\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{u_2}{u_2} x^2} A^{*-1}(\hat{U}_G(\tau)\psi(\kappa)) = \\ &= \frac{1}{\sqrt{u_2}} e^{\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{u_2}{u_2} x^2} A^{*-1}(\hat{U}_G(\tau))A^{*-1}(\psi(\kappa)). \end{aligned} \quad (6.39)$$

To factorize the function ψ and single out the general action of $\hat{U}(t)$, we compute

$$A^{*-1}(\psi(\kappa)) = \psi\left(\frac{x}{u_2}\right) = e^{\log(1/u_2)x \frac{\partial}{\partial x}} \psi(x), \quad (6.40)$$

⁶This fact is of the greatest relevance for the analysis of the inclusion of time-symmetry in [131] for the damped harmonic oscillator.

where $e^{\log(1/u_2)x \frac{\partial}{\partial x}}$ is a dilation operator which is *not* unitary. To unitarize this operator, the generator must be shifted from $x \frac{\partial}{\partial x}$ to $x \frac{\partial}{\partial x} + \frac{1}{2}$, so that the true unitary operator is then

$$\hat{U}_D\left(\frac{1}{u_2}\right) = e^{\log(1/u_2)\left(x \frac{\partial}{\partial x} + \frac{1}{2}\right)} = \frac{1}{\sqrt{u_2}} e^{\log(1/u_2)x \frac{\partial}{\partial x}}. \quad (6.41)$$

But the factor $\frac{1}{\sqrt{u_2}}$ is already present in the previous expression of $\hat{U}(t)$. Therefore, it now reads

$$\begin{aligned} \hat{U}(t) &= e^{\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} A^{*-1} (\hat{U}_G(\tau)) \hat{U}_D\left(\frac{1}{u_2}\right) = \\ &= \frac{1}{\sqrt{u_2}} e^{\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2} e^{\frac{i\hbar}{2m} u_1 u_2 \frac{\partial^2}{\partial x^2}} e^{\log(1/u_2)x \frac{\partial}{\partial x}}. \end{aligned} \quad (6.42)$$

Its inverse is given by

$$\hat{U}(t)^{-1} = \hat{U}(t)^\dagger = \sqrt{u_2} e^{\log(u_2)x \frac{\partial}{\partial x}} e^{\frac{-i\hbar}{2m} u_1 u_2 \frac{\partial^2}{\partial x^2}} e^{-\frac{i}{2} \frac{m}{\hbar} \frac{1}{W} \frac{\dot{u}_2}{u_2} x^2}. \quad (6.43)$$

Interestingly, we have been able to obtain an *exact* expression for the evolution operator as a product of operators. No perturbative approximation method, which could become cumbersome in some cases, is needed for *any* LODE-related quantum system to obtain the evolution operator. These results hold for $\Lambda \neq 0$ (see Subsection 6.1.4).

As a general comment before proceeding with the computation of the wave functions, let us go back to the relevance of conditions (6.14). Those have been chosen to preserve the identity of the variables and basic operators. Any other choice of solutions satisfying different initial conditions at any given initial time would have implications which must be kept under control. This was not taken into account in [118], which might result in some confusing derivations. A general shift

$$u_1 \rightarrow a u_1 + b u_2, \quad u_2 \rightarrow c u_1 + d u_2, \quad (6.44)$$

with the condition $ad - bc = 1$ to preserve the value of the Wronskian W , is equivalent to the canonical transformation in the basic operators

$$\hat{X} \rightarrow a \hat{X} - \frac{b}{m} \hat{P}, \quad \hat{P} \rightarrow -cm \hat{X} + d \hat{P}. \quad (6.45)$$

That is, the freedom in the choice of the solutions u_1 and u_2 , which is a $SL(2, \mathbb{R})$ transformation, stands for a $Sp(1, \mathbb{R})$ transformation in basic operators and defines a family of quantum Arnold transformations. Then, the relation between Hilbert spaces in the lower part of the diagram (6.6) turns into a non-trivial transformation:

$$\hat{A}_0 \psi(x) \equiv \hat{1} \psi(x) \rightarrow \sqrt{d} e^{-\frac{icmx^2}{2\hbar d}} \psi\left(\frac{x}{d}\right). \quad (6.46)$$

Following the general ideas already noted, any linear combination of quadratic operators belonging to the realization of the Schrödinger group (any operator in the subalgebra

of (6.31)-(6.33)) can be chosen in such a way that its eigenfunctions solve (6.12). A specific combination of the operators (6.31)-(6.33) with constant coefficients $\tilde{\omega}$ and $\tilde{\gamma}$ was already chosen in [132]⁷:

$$\hat{H}^* = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \tilde{\omega}^2 \hat{X}^2 + \frac{\tilde{\gamma}}{2} \hat{X} P. \quad (6.47)$$

The eigenfunctions of this operator, solutions of the Schrödinger equation, are

$$\begin{aligned} \phi_\nu(x, t) = & \frac{1}{\sqrt{\sqrt{2\pi}\Gamma(\nu+1)}\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}} e^{\frac{i}{2\hbar} m x^2 \left(\frac{\tilde{\Omega}^2 u_1/(u_2-\tilde{\gamma}u_1/2)}{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2 u_1^2} + \frac{\dot{u}_2-\tilde{\gamma}\dot{u}_1/2}{(u_2-\tilde{\gamma}u_1/2)W} \right)} \\ & \left(\frac{u_2-\tilde{\gamma}u_1/2-i\tilde{\Omega}u_1}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}} \right)^{\nu+\frac{1}{2}} \left(C_1 D_\nu \left(\frac{\sqrt{\frac{2m\tilde{\Omega}}{\hbar}} x}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}} \right) + C_2 D_{-1-\nu} \left(\frac{i\sqrt{\frac{2m\tilde{\Omega}}{\hbar}} x}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}} \right) \right), \end{aligned} \quad (6.48)$$

where C_1 and C_2 are arbitrary constants, D_ν are the parabolic cylinder functions [137], Γ is the Gamma function, $\tilde{\Omega} = \sqrt{\tilde{\omega}^2 - \frac{\tilde{\gamma}^2}{4}}$ and ν is in general a complex number.

In writing $\phi_\nu(x, t)$ we have kept the generality of the quantum Arnold transformation so that these solutions are valid for any LODE-type system (the corresponding formula for a LODE with an external force $\Lambda \neq 0$ is given in the Subsection 6.1.4). This family of wave functions is more general than the one found by Dodonov and Man'ko in [118] in that it contains a set of functions valid when $\frac{\tilde{\gamma}}{2} > \tilde{\omega}$ even for a general LODE system. The associated spectrum of \hat{H}^* is

$$h^* = \hbar \tilde{\Omega} \left(\nu + \frac{1}{2} \right). \quad (6.49)$$

To obtain these solutions, we have taken advantage of the QAT itself performing the following steps. We solve the time-dependent Schrödinger equation corresponding to a harmonic oscillator with frequency $\tilde{\Omega}$, considering both the attractive and the repulsive cases, so that we obtain solutions in terms of parabolic cylinder functions. Then, we take the QAT from this “intermediate” system to the free one. We compose this QAT with the inverse QAT corresponding to passing the free system to the present LODE system, obtaining this way solutions to the LODE system Schrödinger equation, which are eigenfunctions of $\hat{H}_{HO\Omega} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \tilde{\Omega}^2 \hat{X}^2$. Finally, making use of the freedom in the choice in u_1 and u_2 , we perform the shift

$$u_1 \rightarrow u_1, \quad u_2 \rightarrow u_2 - \frac{\tilde{\gamma}}{2} u_1 \quad \Rightarrow \quad \tilde{X} \rightarrow \hat{X}, \quad \tilde{P} \rightarrow \hat{P} + m \frac{\tilde{\gamma}}{2} \hat{X}. \quad (6.50)$$

Its effect on the quadratic operators causes the expression of the particular combination

$$\hat{H}_{HO\Omega} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \tilde{\Omega}^2 \hat{X}^2 \quad \rightarrow \quad \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \tilde{\omega}^2 \hat{X}^2 + \frac{\tilde{\gamma}}{2} \hat{X} P = \hat{H}^* \quad (6.51)$$

⁷For the damped harmonic oscillator with constant ω and γ , coinciding with $\tilde{\omega}$ and $\tilde{\gamma}$ resp., this operator is the only one from the $SL(2, \mathbb{R})$ Schrödinger subalgebra which commutes with the Hamiltonian.

to change. As a consequence, the obtained solutions turn into $\phi_\nu(x, t)$.

The condition of normalizability must be imposed to retain the physically valid solutions. And we observe that the normalizability of the wave functions depends on the specific values of $\tilde{\omega}$ and $\tilde{\gamma}$ in the expression (6.47)⁸. For $\tilde{\omega} > \frac{\tilde{\gamma}}{2}$, the normalizable solutions correspond to $C_2 = 0$ and $\nu = n$ an integer. These functions are written then in terms of the Hermite polynomials⁹. In the case when $\tilde{\Omega}$ is imaginary, the solutions are Dirac-delta normalizable for $\nu = -\frac{1}{2} + i\lambda$, with λ a real number. The operator \hat{H}^* shows a continuous, real, doubly degenerate spectrum in this case [138] (see also [118] for constant ω and γ in the overdamping regime). The critical case $\tilde{\Omega} = 0$ can be obtained as a limit of this case.

It must be emphasized then that the choice of these constants encodes the choice of the particular (arbitrary) quadratic operator belonging to the Schrödinger algebra used to label the solutions. In the framework of the quantum Arnold transformation, this freedom leads to other families of solutions, different from the one presented here.

6.1.3 Dissipative systems: Hamiltonian vs. Hermitian operators

Let us now have a close look at a couple of simple particular cases, extensively studied in the literature: the damped particle and the damped harmonic oscillator. Analogously, it is possible to analyze the harmonic oscillator from the QAT point of view. Although [124] and [118] contain some aspects of this analysis, it is possible to go a bit further and arrive at interesting results like in Section 6.2, where use of the QAT is made to define harmonic oscillator-like states in the free particle and where a higher-dimensional version of the QAT is introduced (see also [135]).

6.1.3.1 Damped particle

For $f = \gamma t$, $\omega = 0$ and $\Lambda = 0$ in (6.10), a Lagrangian for the damped particle can be given:

$$L_{DP} = \frac{1}{2} m e^{\gamma t} \dot{x}^2, \quad (6.52)$$

where γ is the damping constant. The equation of motion is then

$$\ddot{x} + \gamma \dot{x} = 0. \quad (6.53)$$

Two independent solutions for this equation, satisfying initial conditions (6.14),

$$u_1(t) = \frac{1 - e^{-\gamma t}}{\gamma}, \quad u_2(t) = 1, \quad W(t) = e^{-\gamma t}, \quad (6.54)$$

⁸In the specific case of the damped harmonic oscillator, with constant ω and γ , it is possible to identify $\tilde{\omega} \equiv \omega$ and $\tilde{\gamma} \equiv \gamma$ ($\tilde{\Omega} \equiv \Omega$), so that the admissible wave functions vary depending on the regime.

⁹These are the eigenfunctions of the operator \hat{H}^* (there denoted as $\hat{K}(t)$) found in [118] in the general LSODE case. It should be noted that they avoid, in this general case, the explicit mention of dimensional constants equivalent to $\tilde{\omega}$ and $\tilde{\gamma}$ and implicitly entrust the selection of the specific $\hat{K}(t)$ to the choice of the classical solutions, which might become rather confusing.

provide the Arnold transformation for this system:

$$\begin{cases} \tau = \frac{1-e^{-\gamma t}}{\gamma} \\ \kappa = x \\ \varphi = \phi, \end{cases} \quad (6.55)$$

which turns out to be simply a reparametrization in time.

The Schrödinger equation takes the form

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{H}_{DP} \phi \equiv -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2 \phi}{\partial x^2}, \quad (6.56)$$

and the corresponding basic symmetry generators

$$\hat{P} = -i\hbar \frac{\partial}{\partial x}, \quad \hat{X} = x + \frac{i\hbar}{m\gamma} (1 - e^{-\gamma t}) \frac{\partial}{\partial x}. \quad (6.57)$$

The crucial point is to realize that, in fact, the Hamiltonian operator \hat{H}_{DP} does not make sense as an operator acting on the space of solutions of (6.56), while \hat{P} and \hat{X} do. This can be checked by direct computation. For a given solution ϕ , the equation satisfied by $\phi' \equiv \hat{H}_{DP} \phi$ is no longer (6.56), the reason being that \hat{H}_{DP} does not commute with the Schrödinger equation, while $\hat{X} \phi$, for instance, does solve it:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2}{\partial x^2} \right) (\hat{X} \phi) = \left(x + \frac{i\hbar}{m\gamma} (1 - e^{-\gamma t}) \frac{\partial}{\partial x} \right) \left(i\hbar \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2 \phi}{\partial x^2} \right) = 0, \quad (6.58)$$

showing that the Schrödinger equation and \hat{X} do commute.

There is yet another way to check explicitly that \hat{H}_{DP} is ill-defined in the quotient space by the time-evolution generated by itself. Formally, the equation (6.56) can be solved defining a time-evolution operator $\hat{U}(t, t_0)$. The fact that \hat{H}_{DP} commutes at different times,

$$[\hat{H}_{DP}(t_1), \hat{H}_{DP}(t_2)] = 0 \quad (6.59)$$

makes the calculation of $\hat{U}(t, t_0)$ and its action on other operators simple:

$$\hat{U}(t, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^t \hat{H}_{DP}(t') dt'} = e^{\frac{i\hbar}{2m\gamma} (e^{-\gamma t_0} - e^{-\gamma t}) \frac{\partial^2}{\partial x^2}}. \quad (6.60)$$

If we choose $t_0 = 0$, in agreement with conditions (6.14) imposed on solutions (6.54), we recover the Arnold-transformed free evolution operator, as could be expected. However, the computation of the evolution operator directly and the possibility of obtaining it using QAT is not in any way trivial when (6.59) does not hold.

By means of the action of $\hat{U}(t, 0) \equiv \hat{U}(t)$ on the basic operators \hat{P} and \hat{X} , or loosely speaking, using $\hat{U}(t)$ to “de-evolve” them until time $t_0 = 0$, we can show that they match the form (6.19) and determine their action on wave functions depending only on x . This action, in this simple case, can be computed by expanding the exponential evolution operator and performing the commutation operations at each order of the expansion, leading to

$$\hat{U}^\dagger(t) \hat{x} \hat{U}(t) = x, \quad \hat{U}^\dagger(t) \hat{p} \hat{U}(t) = -i\hbar \frac{\partial}{\partial x}, \quad (6.61)$$

which do not depend on time t and take the usual Galilei form¹⁰. This automatically guarantees the three properties mentioned in the previous Section. In contrast, \hat{H}_{DP} does not come down to the quotient by the time evolution generated by itself:

$$\hat{U}^\dagger(t) \hat{H}_{DP} \hat{U}(t) = -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2}{\partial x^2}. \quad (6.62)$$

The reason for \hat{X} and \hat{P} to have good properties is that they are mapped from the free, basic symmetry generators (6.18) and (6.17) by the Arnold transformation (so that they are also symmetry generators of the damped particle system), while \hat{H}_{DP} is not. It is then natural to map one of the quadratic operators belonging to the Schrödinger algebra of the free particle to make it act on the space of quantum solutions of the damped particle. This evolution-like operator defines a proper eigenvalue problem, that can be used to find solutions for (6.56). We could choose the operator \hat{H}^* already mentioned, but in this simple case we prefer to illustrate another rather natural possibility: the free Galilean Hamiltonian

$$\hat{H}_G \equiv \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}. \quad (6.63)$$

In fact, the practical approach would be to solve the free, time-independent Schrödinger equation and Arnold-transform the solutions to obtain (non-stationary) solutions for (6.56). For example, plane waves are mapped into:

$$\phi_k(x, t) = e^{ikx - i\frac{\hbar k^2}{2m\gamma}(1 - e^{-\gamma t})}. \quad (6.64)$$

The observations made above for the damped particle, being quite trivial, can help to clarify the general case.

6.1.3.2 Damped harmonic oscillator

Let us consider a friction function linear in time $f = \gamma t$, as in the case of the damped particle, but also a non-zero constant frequency ω and no external force. The Lagrangian for the Caldirola-Kanai system reads

$$L_{DHO} = e^{\gamma t} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right). \quad (6.65)$$

The classical equation of motion is then

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0. \quad (6.66)$$

Two independent solutions of (6.66), satisfying convenient initial conditions (6.14) are

$$u_1(t) = \frac{1}{\Omega} e^{-\frac{\gamma}{2}t} \sin \Omega t, \quad u_2(t) = e^{-\frac{\gamma}{2}t} \cos \Omega t + \frac{\gamma}{2\Omega} e^{-\frac{\gamma}{2}t} \sin \Omega t, \quad (6.67)$$

¹⁰Note that the “de-evolved” operators take the form of those in (6.57) when $t = 0$. But in general, the correct way to take the quotient by time evolution is that shown in (6.61).

where again $W(t) \equiv \dot{u}_1(t)u_2(t) - u_1(t)\dot{u}_2(t) = e^{-\gamma t}$, and

$$\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}. \quad (6.68)$$

Note that these solutions have good limit in the case of critical damping $\omega = \frac{\gamma}{2}$.

Particularizing the quantum Arnold transformation (6.13) for the free Schrödinger equation (6.7), the Caldirola-Kanai equation is obtained:

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{H}_{DHO} \phi \equiv -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 e^{\gamma t} \phi. \quad (6.69)$$

Basic quantum operators are now given by

$$\hat{P} = -i\hbar \frac{e^{-\frac{\gamma t}{2}}}{2\Omega} (2\Omega \cos \Omega t + \gamma \sin \Omega t) \frac{\partial}{\partial x} + m \frac{e^{\frac{\gamma t}{2}}}{4\Omega} (\gamma^2 + 4\Omega^2) \sin \Omega t x, \quad (6.70)$$

$$\hat{X} = \frac{e^{\frac{\gamma t}{2}}}{2\Omega} (2\Omega \cos \Omega t - \gamma \sin \Omega t) x + i\hbar \frac{e^{-\frac{\gamma t}{2}}}{m\Omega} \sin \Omega t \frac{\partial}{\partial x}. \quad (6.71)$$

It is worth to note that these operators match those that were already found in [118] by hand, in looking for integrals of motion.

Again, the key observation is that \hat{H}_{DHO} does not make sense as an operator acting on the space of solutions of (6.69), while \hat{P} and \hat{X} do respect solutions. Although this can be proved by direct calculation, it is more instructive to obtain the evolution operator $\hat{U}(t, t_0)$. But the fact that \hat{H}_{DHO} does not commute at different times $[\hat{H}_{DHO}(t_1), \hat{H}_{DHO}(t_2)] \neq 0$ makes its calculation trickier using conventional methods, as already mentioned.

Generically, one would approach the problem resorting to a perturbative method. An appropriate method to solve the operator equation for $\hat{U}(t) \equiv \hat{U}(t, 0)$ corresponding to (6.69) (we now make explicit the time dependence of the Hamiltonian),

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}_{DHO}(t) \hat{U}(t), \quad (6.72)$$

is that of the Magnus expansion [136] (see Subsection 6.1.5).

However, it is a good idea to take advantage of the QAT instead. The explicit expression for the *exact* evolution operator encountered for the damped harmonic oscillator is rather involved and is found substituting (6.67) in (6.42):

$$\hat{U}(t) = \sqrt{\frac{2\Omega e^{\frac{\gamma}{2}t}}{2\Omega \cos \Omega t + \gamma \sin \Omega t}} e^{\frac{i}{2} \frac{m}{\hbar} \frac{-2\omega^2 e^{\gamma t} \sin \Omega t}{2\Omega \cos \Omega t + \gamma \sin \Omega t} x^2} e^{\frac{i\hbar}{2m} \frac{1}{2\Omega^2} e^{-\gamma t} \sin \Omega t (2\Omega \cos \Omega t + \gamma \sin \Omega t) \frac{\partial^2}{\partial x^2}} e^{\log\left(\frac{2\Omega e^{\frac{\gamma}{2}t}}{2\Omega \cos \Omega t + \gamma \sin \Omega t}\right) x \frac{\partial}{\partial x}}. \quad (6.73)$$

$\hat{U}^\dagger(t)$ is obtained analogously. It is then possible to check the de-evolution of the operators simply expanding the exponentials to the desired order, just as one would do for the free evolution operator. This allows to state that:

$$\hat{U}^\dagger(t)\hat{X}\hat{U}(t)=x, \quad \hat{U}^\dagger(t)\hat{P}\hat{U}(t)=-i\hbar\frac{\partial}{\partial x}, \quad (6.74)$$

as expected.

The action of $\hat{U}(t)$ on $\hat{H}_{DHO}(t)$ shows that it does not fall down to the quotient by the time evolution generated by itself. In fact, the de-evolution,

$$\begin{aligned} \hat{U}^\dagger(t)\hat{H}_{DHO}\hat{U}(t)= \\ \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2\right)-\gamma t\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{1}{2}m\omega^2x^2\right) \\ +\frac{\gamma^2t^2}{2}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2+\frac{2\omega^2}{\gamma}\left(-i\hbar x\frac{\partial}{\partial x}-i\hbar\right)\right)+\mathcal{O}(t^3), \end{aligned} \quad (6.75)$$

does depend on time at any order in the time expansion.

We remark once again that there is no actual need for a perturbative method to obtain $\hat{U}(t)$ in this case when QAT is used.

The computation of the wave functions, solutions of (6.69), can follow the steps shown in Section 6.1.2. We select the operator \hat{H}^* particularized for the damped harmonic oscillator, that is, substituting u_1 and u_2 by (6.67) and identifying $\tilde{\omega} \equiv \omega$ and $\tilde{\gamma} \equiv \gamma$. Its general eigenfunctions are given by the corresponding expression (6.48), and the spectrum (6.49) of \hat{H}^* will depend on the regime fixed by the specific value of Ω .

6.1.4 Inhomogeneous LODE

We give here the general Arnold transformation with an extra external force term Λ and the corresponding generalization of the main results above. This computation follows analogous steps as those shown before.

The general QAT is given by

$$\begin{cases} \tau = \frac{u_1(t)}{u_2(t)} \\ \kappa = \frac{x-u_p(t)}{u_2(t)} \\ \varphi = \phi \sqrt{u_2(t)} e^{-\frac{i}{\hbar} \frac{m}{w(t)} \frac{\dot{u}_2(t)}{u_2(t)} (x-u_p(t))^2 - i \frac{m}{\hbar} \frac{1}{w(t)} \dot{u}_p(t)x - \frac{i}{\hbar} \frac{m}{w(t)} \int \frac{1}{w(t)} (u_p(t)^2 \omega(t)^2 - \dot{u}_p^2) dt} \end{cases} \quad (6.76)$$

The extra conditions to be imposed on the classical solution $u_p(t)$ to preserve the identity of x and t before and after the transformation are:

$$u_p(0)=0, \quad \dot{u}_p(0)=0. \quad (6.77)$$

In fact, the solution $u_p(t)$ can be expressed:

$$u_p(t)=K_1(t)u_1(t)+K_2(t)u_2(t), \quad (6.78)$$

where:

$$K_1(t) = \int_0^t \frac{u_2(t')}{W(t')} \Lambda(t') dt', \quad K_2(t) = - \int_0^t \frac{u_1(t')}{W(t')} \Lambda(t') dt', \quad (6.79)$$

The transformation (6.76) leads to the expressions for basic operators:

$$\begin{aligned} \hat{P} &= -i\hbar u_2 \frac{\partial}{\partial x} - m \frac{\dot{u}_2}{W} (x - u_p) - m \frac{u_2}{W} \dot{u}_p \\ \hat{X} &= \frac{\dot{u}_1}{W} (x - u_p) + \frac{u_1}{W} \dot{u}_p + \frac{i\hbar}{m} u_1 \frac{\partial}{\partial x}. \end{aligned} \quad (6.80)$$

The evolution operator reads

$$\begin{aligned} \hat{U}(t) &= \frac{1}{\sqrt{u_2}} e^{\frac{i}{\hbar} m \frac{1}{W} \frac{\dot{u}_2}{u_2} (x - u_p)^2 + i \frac{m}{\hbar} \frac{1}{W} \dot{u}_p x + \frac{i}{\hbar} \frac{m}{W} \int \frac{1}{W} (u_p^2 \omega(t)^2 - \dot{u}_p^2) dt} \\ & e^{\frac{i\hbar}{2m} u_1 u_2 \frac{\partial^2}{\partial x^2}} e^{-u_p \frac{\partial}{\partial x}} e^{\log(1/u_2) x \frac{\partial}{\partial x}}. \end{aligned} \quad (6.81)$$

Finally, the general solution of the corresponding Schrödinger equation, eigenfunction of the operator \hat{H}^* , is:

$$\begin{aligned} \phi_\nu(x, t) &= \frac{1}{\sqrt{\sqrt{2\pi}\Gamma(\nu+1)} \sqrt{(u_2 - \tilde{\gamma} u_1/2)^2 + \tilde{\Omega}^2 u_1^2}} e^{\left(\frac{i}{2\hbar} m (x - u_p)^2 \frac{\tilde{\Omega}^2 u_1 / (u_2 - \tilde{\gamma} u_1/2)}{(u_2 - \tilde{\gamma} u_1/2)^2 + \tilde{\Omega}^2 u_1^2} + \frac{i}{2\hbar} m x^2 \frac{\dot{u}_2 - \tilde{\gamma} \dot{u}_1/2}{(u_2 - \tilde{\gamma} u_1/2) W} \right)} \\ & e^{\left(\frac{im}{2\hbar} \int \frac{1}{W} (u_p^2 \omega(t)^2 - \dot{u}_p^2) dt - \frac{imx u_p (\dot{u}_2 - \tilde{\gamma} \dot{u}_1/2)}{\hbar W (u_2 - \tilde{\gamma} u_1/2)} + \frac{im u_p^2 (\dot{u}_2 - \tilde{\gamma} \dot{u}_1/2)}{2\hbar W (u_2 - \tilde{\gamma} u_1/2)} + \frac{imx \dot{u}_p}{\hbar W} \right)} \\ & \left(\frac{u_2 - \tilde{\gamma} u_1/2 - i\tilde{\Omega} u_1}{\sqrt{(u_2 - \tilde{\gamma} u_1/2)^2 + \tilde{\Omega}^2 u_1^2}} \right)^{\nu + \frac{1}{2}} \left(C_1 D_\nu \left(\frac{\sqrt{\frac{2m\tilde{\Omega}}{\hbar}} (x - u_p)}{\sqrt{(u_2 - \tilde{\gamma} u_1/2)^2 + \tilde{\Omega}^2 u_1^2}} \right) + C_2 D_{-1-\nu} \left(\frac{i\sqrt{\frac{2m\tilde{\Omega}}{\hbar}} (x - u_p)}{\sqrt{(u_2 - \tilde{\gamma} u_1/2)^2 + \tilde{\Omega}^2 u_1^2}} \right) \right). \end{aligned} \quad (6.82)$$

6.1.5 Appendix: The Magnus expansion

The Magnus expansion was introduced as a tool to solve non-autonomous linear differential equations for linear operators and has the very attractive property of leading to approximate solutions which exhibit unitarity at any order of approximation. This is in contrast to the representation in terms of the time-ordering operator \mathcal{T} introduced by Dyson.

A solution to (6.72) is given by

$$\hat{U}(t) = e^{\hat{\Omega}(t)}, \quad \hat{\Omega}(0) = 0, \quad (6.83)$$

and a series expansion for the matrix in the exponent

$$\hat{\Omega}(t) = \sum_{k=1}^{\infty} \hat{\Omega}_k(t) \quad (6.84)$$

which is called the Magnus expansion. We can write down the first three terms of that series:

$$\hat{\Omega}_1(t) = \int_0^t dt_1 \left(-\frac{i}{\hbar} \hat{H}_{DHO}(t_1) \right) \quad (6.85)$$

$$\hat{\Omega}_2(t) = \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \left[-\frac{i}{\hbar} \hat{H}_{DHO}(t_1), -\frac{i}{\hbar} \hat{H}_{DHO}(t_2) \right] \quad (6.86)$$

$$\begin{aligned} \hat{\Omega}_3(t) = \frac{1}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 & \left(\left[-\frac{i}{\hbar} \hat{H}_{DHO}(t_1), \left[-\frac{i}{\hbar} \hat{H}_{DHO}(t_2), -\frac{i}{\hbar} \hat{H}_{DHO}(t_3) \right] \right] \right. \\ & \left. + \left[-\frac{i}{\hbar} \hat{H}_{DHO}(t_3), \left[-\frac{i}{\hbar} \hat{H}_{DHO}(t_2), -\frac{i}{\hbar} \hat{H}_{DHO}(t_1) \right] \right] \right). \end{aligned} \quad (6.87)$$

However, a good iterative method to obtain the operator $\hat{\Omega}(t)$ is given by the formula:

$$\hat{\Omega}^{[n]}(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \int_0^t dt_1 \text{ad}_{\hat{\Omega}^{[n-1]}(t_1)}^k \left(-\frac{i}{\hbar} \hat{H}_{DHO}(t_1) \right) \quad (6.88)$$

$$\hat{\Omega}(t) = \lim_{n \rightarrow \infty} \hat{\Omega}^{[n]}(t), \quad (6.89)$$

where B_k are the Bernoulli numbers and

$$\text{ad}_{\hat{A}}^0(\hat{B}) \equiv \hat{B}, \quad \text{ad}_{\hat{A}}^1(\hat{B}) \equiv [\hat{A}, \hat{B}], \quad \text{ad}_{\hat{A}}^k(\hat{B}) \equiv [\text{ad}_{\hat{A}}^{k-1}(\hat{B}), \hat{B}]. \quad (6.90)$$

We have computed the operator $\hat{\Omega}(t)$ for the case of the damped harmonic oscillator to sixth order of approximation, to give:

$$\begin{aligned} \hat{\Omega}^{[6]}(t) = & \\ & -\frac{i}{\hbar} t \left(\left(1 + \frac{\gamma^2 t^2}{6} + \frac{\gamma^4 t^4}{120} \left(1 + \frac{2\omega^2}{\gamma^2} \right) + \frac{\gamma^6 t^6}{5040} \left(1 + \frac{16\omega^2}{\gamma^2} + \frac{32\omega^4}{3\gamma^4} \right) \right) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) \right. \\ & -\frac{\gamma t}{2} \left(1 + \frac{\gamma^2 t^2}{12} + \frac{\gamma^4 t^4}{360} \left(1 + \frac{6\omega^2}{\gamma^2} \right) \right) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{1}{2} m \omega^2 x^2 \right) \\ & \left. + \frac{\gamma \omega^2 t}{6} \left(1 + \frac{\gamma^2 t^2}{20} \left(1 + \frac{4\omega^2}{3\gamma^2} \right) + \frac{\gamma^4 t^4}{840} \left(1 + \frac{44\omega^2}{3\gamma^2} + \frac{16\omega^4}{3\gamma^4} \right) \right) \left(-i\hbar x \frac{\partial}{\partial x} - \frac{1}{2} i\hbar \right) \right) \quad (6.91) \end{aligned}$$

With this approximation to $\hat{\Omega}(t)$, obtained by means of (6.88), one can obtain the quotient by time evolution of a certain operator $\hat{O}(t)$, given by:

$$\hat{O} = e^{-\hat{\Omega}(t)} \hat{O}(t) e^{\hat{\Omega}(t)} = \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_{-\hat{\Omega}(t)}^k (\hat{O}(t)). \quad (6.92)$$

These formulas lead to the same results found in Subsection 6.1.3.2.

6.2 Humps

In the context of the quantum free particle, the eigenstates of the Hamiltonian, which are also eigenstates of the momentum operator, are not normalizable. These states, the plane waves, are fully delocalized. However, it is customary to expand any normalizable solution of the free Schrödinger equation in terms of plane waves, using the Fourier transform, building in this way wave packets which represent localized solutions. The simplest example is the Gaussian wave packet, which has the property of minimizing the uncertainty relations between the position and the momentum operator.

In this section we construct discrete basis of the space of solutions of the quantum free particle which are the map through the QAT of the eigenstates of the quantum harmonic oscillator. The first states of these basis are Gaussian wave packets. They are not eigenstates of the free particle Hamiltonian, i.e. they are not stationary states, but, rather, eigenfunctions of a certain quadratic operator \hat{N} with discrete eigenvalues. The lowest ones, the Gaussian wave packets, are localized with arbitrary initial size, which is related to the oscillator frequencies and can be conveniently chosen, and have initial minimal uncertainties. The following ones are “multi-localized” in the sense that, for instance, in one dimension the n^{th} -order state presents n zeros and $n + 1$ humps which spread out with time. This mimics the situation for the harmonic oscillator to such an extent that, in one dimension, it is possible to build creation and annihilation operators \hat{a}^\dagger and \hat{a} . The number operator is going to be $\hat{N} \sim \hat{a}^\dagger \hat{a}$, although in the case of the free particle it is not the Hamiltonian. Going even further, we give a set of “coherent states” which are interpreted as traveling wave packets. This construction can be easily generalized to higher dimensions in different coordinate systems.

This construction can be of physical relevance in Quantum Information Theory, using these states to transmit information, as well as in Atom Optics and Quantum Computation. It might also be useful in describing scattering process in a discrete basis, instead of using plane waves.

6.2.1 A discrete basis of wave packets

Let \mathcal{H} be the Hilbert space of solutions of the free particle Schrödinger equation, and \mathcal{H}_{HO} the one corresponding to the Harmonic oscillator. Since QAT involves a change of variables, we shall denote by $\psi(x, t) \in \mathcal{H}$ the free particle solutions and by $\varphi(x', t') \in \mathcal{H}_{\text{HO}}$ the harmonic oscillator ones.

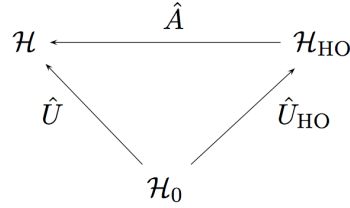
For the case of the harmonic oscillator $\dot{f} = 0$ and $\omega(t') = \omega$, and the two independent solutions that define the QAT can be chosen as $u_1(t') = \frac{1}{\omega} \sin(\omega t')$ and $u_2(t') = \cos(\omega t')$,

with $W(t') = 1$. It can be checked that the change of variables results in:

$$t' = \frac{1}{\omega} \arctan(\omega t)$$

$$x' = \cos(\arctan(\omega t))x = \frac{x}{\sqrt{1 + \omega^2 t^2}}. \quad (6.93)$$

Diagrammatically, the QAT can be represented as:



where \hat{U} and \hat{U}_{HO} stand for the evolution operators for the and harmonic oscillators, respectively, while \mathcal{H}_0 is the Hilbert space, either for the free particle and harmonic oscillator, of solutions of their respective Schrödinger equations at $t = 0$ (see Section 6.1 for the conditions under which \mathcal{H}_0 is common to both systems; see also [140]).

Thanks to the commutativity of this diagram, and the unitarity of the operators appearing in it, we can map objects (wavefunctions, operators, expectation values, uncertainties) from one system to the other.

Applying now the QAT to the time-dependent harmonic oscillator eigenstates,

$$\varphi_n(x', t') = \frac{\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}}{\sqrt{2^n n!}} e^{-i\omega(n+\frac{1}{2})t'} e^{-\frac{m\omega}{2\hbar}x'^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right), \quad (6.94)$$

we obtain the following sets of states, solutions of the Schrödinger equation for the free particle:

$$\psi_n(x, t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L |\delta|}} e^{-\frac{x^2}{4L^2 \delta}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n\left(\frac{x}{\sqrt{2L|\delta|}}\right), \quad (6.95)$$

where, in order to obtain a more compact notation, we have introduced the quantities $L = \sqrt{\frac{\hbar}{2m\omega}}$, with dimensions of length, and $\tau = \frac{2mL^2}{\hbar} = \omega^{-1}$, with dimensions of time. We also denote by δ the complex, time dependent, adimensional expression $\delta = 1 + i\omega t = 1 + i\frac{\hbar t}{2mL^2} = 1 + it/\tau$.

The fact that these states are written in terms of the Hermite polynomials can be used to show that the set of states is a basis for the Hilbert space of solutions of the free Schrödinger equation, $L^2(\mathbb{R})$. In fact, at $t = 0$, $\psi_n(x, 0)$ are the Hermite functions, which constitute a basis of $L^2(\mathbb{R})$. Since the time evolution is unitary, the set of states $\psi_n(x, t)$ is still a basis for any time t .

The family of wavefunctions (6.95) has been known in the literature as Hermite-Gauss wave packets [141], and they have been widely used, in their two dimensional version (see

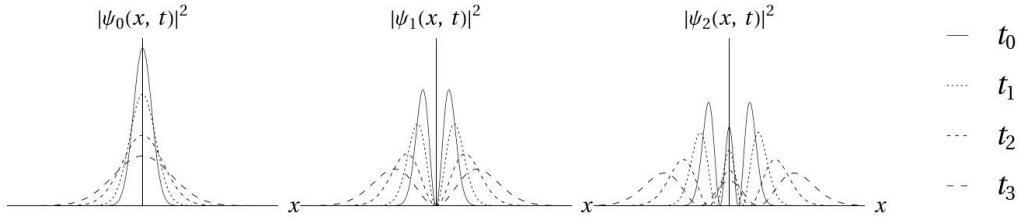


Figure 6.1: Spreading under time evolution of wave functions ψ_0 , ψ_1 and ψ_2 , with $t_k = k\tau$.

Section 6.2.3), in paraxial wave optics [142]¹¹. However, this kind of states and the ones constructed in Section 6.2.3 are better understood in the framework of the QAT.

The first state of this basis, the one mapped from the harmonic oscillator vacuum state, is given by:

$$\psi_0(x, t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|} \right)^{\frac{1}{2}} e^{-\frac{x^2}{4L^2\delta}} = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L\delta}} e^{-\frac{x^2}{4L^2\delta}}, \quad (6.96)$$

which is nothing other than a Gaussian wave packet with center at the origin and width L . The parameter τ is the dispersion time of the Gaussian wave packet, (see, for instance, [1])

Figure 6.1 shows some of these wave functions and how they evolve in time.

We see that the number of “parts”, or *humps*, of the wave functions, determined by the number of zeros, is quantized, in the sense that there is one hump between two consecutive zeros. This property will be important for the physical applications discussed at the end of this subsection.

The QAT also allows to map operators from one Hilbert space to the other (see Section 6.1 and [140]), in such a way that ladder operators for the Harmonic oscillator can be mapped to ladder operators for the free particle that act as creation and annihilation operators for these states:

$$\begin{aligned} \hat{a} &= L\delta \frac{\partial}{\partial x} + \frac{x}{2L} \\ \hat{a}^\dagger &= -L\delta^* \frac{\partial}{\partial x} + \frac{x}{2L}. \end{aligned} \quad (6.97)$$

The action of \hat{a} and \hat{a}^\dagger on these wave functions is the usual one:

$$\hat{a}\psi_n = \sqrt{n}\psi_{n-1}, \quad \hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}. \quad (6.98)$$

It is possible to introduce this discrete basis without resorting to the QAT, in a very intuitive manner. The key point is that the operator \hat{a} annihilates the Gaussian wave packet,

¹¹In [118] a basis of discrete states and their corresponding coherent states were constructed for the damped harmonic oscillator, which, in a certain limiting process would reproduce ours.

and this characterizes it. The whole family of states (6.95) can be generated acting with the adjoint operator \hat{a}^\dagger of \hat{a} . The rest of the construction, i.e. coherent states, uncertainties, etc. proceeds without the need of the QAT. However, as we shall see below, the QAT is very useful when computing difficult things in a very easy way.

A very useful property of the QAT is its unitarity, implying that it preserves scalar products and therefore expectation values. Denoting by $\langle \cdot, \cdot \rangle^f$ the scalar product in the Hilbert space of solutions of the free particle Schrödinger equation, and by $\langle \cdot, \cdot \rangle^o$ the corresponding one in the harmonic oscillator Hilbert space, we have that:

$$\langle \psi^f, \phi^f \rangle^f = \langle \psi^o, \phi^o \rangle^o \quad \langle \hat{O}^f \rangle_{\psi^f}^f = \langle \hat{O}^o \rangle_{\psi^o}^o \quad (6.99)$$

where ψ^f, ϕ^f and \hat{O}^f are related to ψ^o, ϕ^o and \hat{O}^o through the QAT, respectively. It is important to note (see Section 6.1) that \hat{O}^f and \hat{O}^o are invariant operators (also known as constants or integral of motion operators) with respect to their respective temporal evolution, and preserve their respective Hilbert spaces. These operators are, essentially, those closing a Heisenberg-Weyl algebra and their powers. Among them, the most important ones are those generating the Schrödinger group (see below).

Another aspect of this quantum realization is shown when computing the uncertainties associated with each wave function. As a function of time, for each state ψ_n , they read:

$$\Delta \hat{x}_n \Delta \hat{p}_n = (n + \frac{1}{2}) \hbar |\delta| \quad (6.100)$$

where $\Delta \hat{x}_n = \sqrt{\langle (\hat{x}^f)^2 \rangle_{\psi_n^f}^f - (\langle \hat{x}^f \rangle_{\psi_n^f}^f)^2}$ and $\Delta \hat{p}_n = \sqrt{\langle (\hat{p}^f)^2 \rangle_{\psi_n^f}^f - (\langle \hat{p}^f \rangle_{\psi_n^f}^f)^2}$ have been computed using that¹²:

$$\begin{aligned} \langle (\hat{x}^f)^2 \rangle_{\psi_n^f}^f &= \frac{1}{u_2(t')^2} \langle (\hat{x}^o)^2 \rangle_{\psi_n^o}^o = |\delta|^2 \frac{\hbar}{m\omega} (n + \frac{1}{2}) \\ \langle (\hat{p}^f)^2 \rangle_{\psi_n^f}^f &= \langle (\hat{p}^o)^2 \rangle_{\psi_n^o}^o = m\omega\hbar (n + \frac{1}{2}). \end{aligned} \quad (6.101)$$

To compute these expectations values using the QAT, we have used that $\hat{x}^f \equiv \hat{x} = \hat{X} + \frac{t}{m} \hat{P}$, where \hat{X} and \hat{P} are constants of motion operators for the free particle.

For $n = 0$ the time evolution of the uncertainty is the one which results from the usual Gaussian wave packet [1], and, among all, the minimal one.

The number operator associated with the creation and annihilation operators above will provide the position of the state in this grid of uncertainties. We can compute it (or map it from the number operator for the Harmonic oscillator) in the usual way:

$$\hat{N} = \frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \left[-|\delta|^2 L^2 \frac{\partial^2}{\partial x^2} + i \frac{t}{\tau} (x \frac{\partial}{\partial x} + \frac{1}{2}) + \frac{x^2}{4L^2} \right]. \quad (6.102)$$

By making use of the Schrödinger equation, we can turn this operator into a first-order one:

$$\hat{N} = \left[i|\delta|^2 \tau \frac{\partial}{\partial t} + i \frac{t}{\tau} (x \frac{\partial}{\partial x} + \frac{1}{2}) + \frac{x^2}{4L^2} \right], \quad (6.103)$$

¹²In the last equality of the equations the quantum Virial Theorem for the harmonic oscillator potential [1], which is homogeneous of degree two, has been used.

this expression being valid only on solutions of the Schrödinger equation. The action of this operator is such that:

$$\hat{N}\psi_n(x, t) = \left(n + \frac{1}{2}\right)\psi_n(x, t), \quad (6.104)$$

thus reproducing the uncertainties (up to \hbar) given in Eq. (6.100) at time $t = 0$.

It is quite interesting to note that this operator belongs to the Lie algebra of the Schrödinger group, builded up with constants of motion operators up to quadratic order (see Sec. 6.1). It is easily checked that \hat{N} is in this Lie algebra, its relation with the basis above being:

$$\hat{N} = \frac{1}{\hbar\omega} \left(\frac{1}{2m} \hat{P}^2 + \frac{m\omega}{2} \hat{X}^2 \right) = \frac{1}{\hbar\omega} \hat{H}_{\text{HO}}, \quad (6.105)$$

where \hat{H}_{HO} is the operator corresponding to an harmonic oscillator of frequency $\omega = \frac{\hbar}{mL^2} = \frac{1}{\tau}$, but written in terms of constants of motion of the free particle. See Section 6.1 for the relevance of quadratic operators like \hat{N} , but distinct from the Hamiltonian, for building basis of the Hilbert space (see also [117, 118, 132]).

6.2.2 Coherent states or traveling wave packets

As a natural consequence of the introduction of creation and annihilation operators, we construct a set of coherent states for the free particle as the eigenstates of the annihilation operator (they could also be obtained from the usual harmonic oscillator coherent states through the QAT). These states are of the form

$$\phi_a(x, t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|} \right)^{\frac{1}{2}} e^{-\frac{(x-x_0)^2 + x_0\nu_0 t + i\tau\nu_0(\nu_0 t - 2x + x_0)}{4L^2\delta}}, \quad (6.106)$$

where a is the complex number

$$a = \frac{x_0}{2L} + i \frac{m\nu_0 L}{\hbar} = \frac{1}{2L}(x_0 + i\nu_0\tau), \quad (6.107)$$

and they verify:

$$\hat{a} \phi_a(x, t) = a \phi_a(x, t). \quad (6.108)$$

These states can also be obtained by the action of a Galilean boost with parameter ν_0 and a translation by x_0 on the vacuum Gaussian packet, and they constitute an over-complete set of the Hilbert space of the free particle. Coherent states represent traveling Gaussian wave packets, with mean momentum and initial position $m\nu_0$ and x_0 , respectively. They are not eigenstates of the number operator, but its expectation values on these states are:

$$\langle \phi_a | \hat{N} | \phi_a \rangle = \hbar(|a|^2 + \frac{1}{2}), \quad (6.109)$$

where use of (6.99) and (6.105) has been done. Acting by Galilean boosts and translations on a fixed state of the basis, $\psi_n(x, t)$, a new over-complete set of states is obtained, with elements:

$$\phi_a^n(x, t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L |\delta|}} \left(\frac{\delta^*}{|\delta|} \right)^{n+\frac{1}{2}} H_n \left(\frac{x - x_0 - v_0 t}{\sqrt{2L |\delta|}} \right) e^{-\frac{(x - x_0)^2 + x_0 v_0 t + i \tau v_0 (v_0 t - 2x + x_0)}{4L^2 \delta}}, \quad (6.110)$$

representing traveling multi-localized wave packets bearing $n + 1$ humps, with mean momentum and initial position $m v_0$ and x_0 , respectively, where a is given by (6.107). The fact that these sets are over-complete is a general property, for $t = 0$, of coherent states of the Heisenberg-Weyl group (see [144]). Since the time evolution is unitary, this property is kept at any t . As in the case $n = 0$, these states are not eigenstates of the number operator, but the expectation values, computed as before, are:

$$\langle \phi_a^n | \hat{N} | \phi_a^n \rangle = \hbar (|a|^2 + n + \frac{1}{2}). \quad (6.111)$$

Being a set of coherent states, the uncertainty relations of $\phi_a^n, \forall a \in \mathbb{C}$, are the same as those of ϕ_n given in Eq. (6.100) (see [144]).

6.2.3 Discrete basis of wave packets in higher dimensions

The generalization of the previous construction to higher dimensions is immediate. In more dimensions, due to the symmetries of the harmonic oscillator, we can choose different separation of variables to solve the Schrödinger equation for the harmonic oscillator, which amounts to find simultaneous eigenstates for the harmonic oscillator hamiltonian and other operators commuting with it and among themselves. For instance, separation of variables in cartesian coordinates in N -dimensions is equivalent to diagonalizing simultaneously the harmonic oscillator hamiltonian, and the 1D harmonic oscillator in each coordinate. This can always be done even if the frequencies for each direction are different. The common basis of eigenstates is the product of 1D harmonic oscillator eigenstates in each variable. If we impose spherical symmetry, then all frequencies must coincide and we can search for common eigenstates of the Hamiltonian and the angular momentum operators commuting with it and among themselves (for instance, in three dimensions they would be \hat{L}^2 and \hat{L}_z , and in two dimensions it would be only \hat{L}).

For the case of Cartesian coordinates, the generalization of the previous sections is immediate, the discrete basis of states for the N -dimensions free particle being products of N copies of (6.95) in each of the N variables (x_1, x_2, \dots, x_N) . If $\psi_{n_i}(x_i, t)$ represent a wave packet of the form given in (6.95) but in the variable x_i and with width L_i (and with the corresponding frequency ω_i), then the N -dimensional wave packets are:

$$\psi_{\vec{n}}(\vec{x}, t) = \prod_{i=1}^N \psi_{n_i}(x_i, t), \quad (6.112)$$

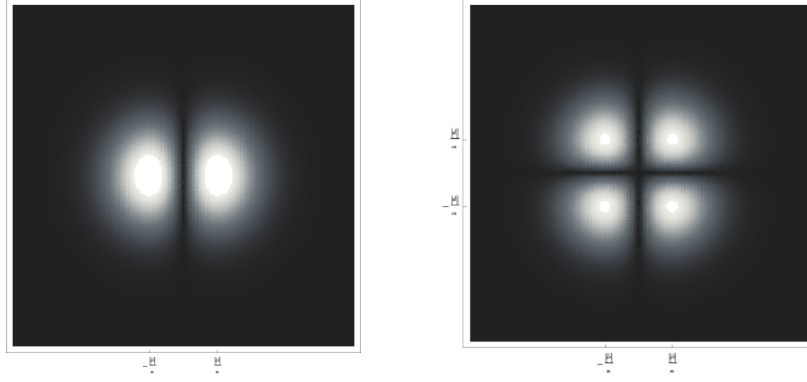


Figure 6.2: Density plot of the probability distribution for the Hermite-Gauss states $\psi_{(1,0)}$ and $\psi_{(1,1)}$. The position of the maxima is shown.

where $\vec{n} = (n_1, n_2, \dots, n_N)$. If all the widths $L_i = L$ are identical, it can be written as:

$$\psi_{\vec{n}}(\vec{x}, t) = \frac{(2\pi)^{-\frac{N}{4}}}{(L|\delta|)^{N/2}} e^{-\frac{\sum_{i=1}^N x_i^2}{4L^2\delta}} \left(\frac{\delta^*}{|\delta|}\right)^{\sum_{i=1}^N n_i + \frac{N}{2}} \prod_{i=1}^N \frac{1}{\sqrt{2^{n_i} n_i!}} H_{n_i}\left(\frac{x_i}{\sqrt{2L|\delta|}}\right), \quad (6.113)$$

Coherent states, or traveling wave packets, are defined similarly, as products of N copies of (6.110), with a vector $\vec{a} \in \mathbb{C}^N$:

$$\phi_{\vec{a}}^{\vec{n}}(\vec{x}, t) = \prod_{i=1}^N \phi_{a_i}^{n_i}(x_i, t). \quad (6.114)$$

For $N = 2$ these states have been widely used in the paraxial approximation to the Helmholtz equation of wave optics, known as the Hermite-Gauss states [142]. Due to the analogy of the paraxial approximation to the Helmholtz equation (with the z coordinate acting as time) and Schrödinger equation in two dimensions, they have been also exploited in atom optics and matter waves [143].

In the case of cylindrical symmetry in paraxial wave optics, the polar version of these states have been used, known as Laguerre-Gauss states [142]. In Figure 6.2 a density plot of the probability distribution for the Hermite-Gauss states $\psi_{(1,0)}$ and $\psi_{(1,1)}$ are shown. The corresponding states for the free Schrödinger equation in two dimensions are:

$$\psi_{n,l}^{\pm}(r, \phi, t) = \sqrt{\frac{n!}{2\pi\Gamma(n+l+1)L^2|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{2n+l+1} e^{\pm il\phi} e^{-\frac{r^2}{4L^2\delta}} \left(\frac{r}{\sqrt{2L|\delta|}}\right)^l L_n^l\left(\frac{r^2}{2L^2|\delta|^2}\right), \quad (6.115)$$

where $n, l = 0, 1, 2, \dots$, and $L_n^l(x)$ are Laguerre polynomials. The state with $n = 0, l = 0$ is the Gaussian wave packet in two dimensions.

These states are eigenstates of the angular momentum operator \hat{L} in 2 dimensions, with values $\hat{L}\psi_{n,l}^{\pm}(r, \phi, t) = \pm l\psi_{n,l}^{\pm}(r, \phi, t)$. In Figure 6.3 a density plot of the probability distribution for the Laguerre-Gauss states $\psi_{0,1}^{\pm}$ and $\psi_{1,1}^{\pm}$ are shown.

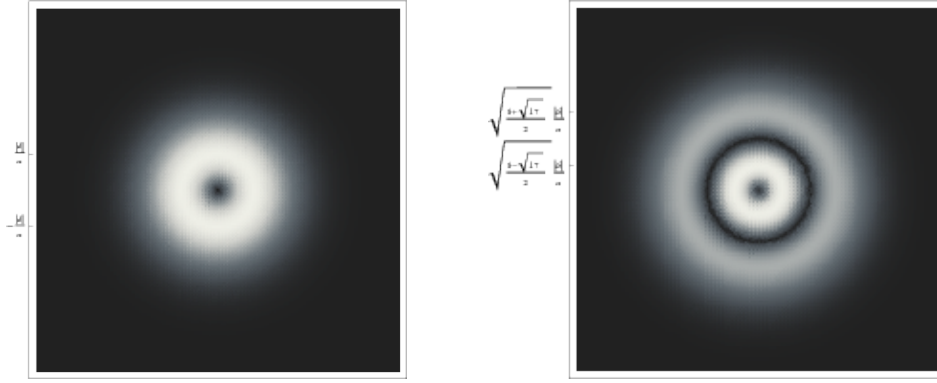


Figure 6.3: Density plot of the probability distribution for the Laguerre-Gauss states $\psi_{0,1}^{\pm}$ and $\psi_{1,1}^{\pm}$. The position of the maxima is shown.

The generalization to three dimensions in spherical coordinates is straightforward, the states having the form of (6.115) but in terms of spherical harmonics and confluent hypergeometric functions $M(\cdot, \cdot; \cdot)$ (which are also polynomials in this case):

$$\psi_{n,l,m}(r, \theta, \phi, t) = \sqrt{\frac{\Gamma(l+3/2+n-1)}{\sqrt{2}(n-1)!\Gamma(l+3/2)^2 L^3 |\delta|}} \left(\frac{\delta^*}{|\delta|} \right)^{2(n-1)+l+3/2} e^{-\frac{r^2}{4L^2\delta}}$$

$$Y_l^m(\theta, \phi) \left(\frac{r}{\sqrt{2}L|\delta|} \right)^l M(-n+1, l+3/2; \frac{r^2}{2L^2|\delta|^2}), \quad (6.116)$$

None of these states are eigenstates of the free particle Hamiltonian, but can be seen to have expectation values of the energy equal to half the energy of the corresponding, through the QAT, harmonic oscillator eigenstates¹³:

$$\begin{aligned} \langle \hat{H} \rangle_{\psi_{\vec{n}}} &= \frac{1}{2} E_{\vec{n}} & E_{\vec{n}} &= \hbar\omega \left(\sum_{i=1}^N n_i + \frac{N}{2} \right) \\ \langle \hat{H} \rangle_{\psi_{n,l}^{\pm}} &= \frac{1}{2} E_{nl} & E_{nl} &= \hbar\omega (2n + l + 1) \\ \langle \hat{H} \rangle_{\psi_{n,l,m}} &= \frac{1}{2} E_{nlm} & E_{nlm} &= \hbar\omega (2(n-1) + l + \frac{3}{2}) \end{aligned} \quad (6.117)$$

These states, as in the one dimensional case, have many humps. For Cartesian coordinates they have $\prod_{i=1}^N (n_i + 1)$ humps, while in polar coordinates they have $(n + 1)$ humps (seen in the radial coordinate) with annular form.

Uncertainty relations can also be computed for these states. However, only the cartesian version of the operators \vec{x} and \vec{p} can be used since the canonical momentum associated to the radial coordinate, p_r , it is not self-adjoint. The expressions, computed as in

¹³This a consequence, again, of (6.99) and the quantum Virial theorem for the quadratic, homogeneous harmonic oscillator potential.

the one dimensional case, are:

$$(\Delta x_i)_{\psi_{\vec{n}}}(\Delta p_i)_{\psi_{\vec{n}}} = \frac{1}{\omega} |\delta| E_{n_i}, \quad i = 1, \dots, N$$

$$(\Delta x)_{\psi_{nl}}(\Delta p_x)_{\psi_{nl}} = (\Delta y)_{\psi_{nl}}(\Delta p_y)_{\psi_{nl}} = \frac{1}{2\omega} |\delta| E_{nl} \quad (6.118)$$

$$(\Delta x)_{\psi_{nlm}}(\Delta p_x)_{\psi_{nlm}} = (\Delta y)_{\psi_{nlm}}(\Delta p_y)_{\psi_{nlm}} = (\Delta z)_{\psi_{nlm}}(\Delta p_z)_{\psi_{nlm}} = \frac{1}{3\omega} |\delta| E_{nlm}$$

6.2.4 Physical applications

The theoretical relevance of these multi-localized traveling wave packets, as discrete basis of normalized free particle states, is of no doubt. Similar constructions, like the Harmonic Oscillator (HO) method [145] or the Transformed Harmonic Oscillator (THO) method [146] have been proposed, mainly in nuclear physics, to describe the bound states and the continuum spectrum in a discrete basis. But there the construction is a mathematical tool for approximating the solutions, with no physical meaning. Our states, however, are physically meaningful (as traveling wave packets) and experimentally feasible (see next subsection).

Among the possible theoretical applications, we could think of expanding plane waves in terms of the discrete basis $\{\phi_{\vec{a}}^{\vec{n}}\}_{\vec{n} \in \mathbb{N}_o^N}$, with fixed $\vec{a} \in \mathbb{C}^N$, and describing scattering process in a discrete basis, or expanding arbitrary wave packets in a continuum over-complete set $\{\phi_{\vec{a}}^{\vec{n}}\}_{\vec{a} \in \mathbb{C}^N}$, with fixed $\vec{n} \in \mathbb{N}_o^N$, which could be discretized in a lattice $\mathbb{Z}^N \times \mathbb{Z}^N$ of points while keeping the over-complete character (they are over-complete for $t = 0$ if the volume of the unit cell is smaller than \hbar^N , and again by the unitary time evolution they continue to be over-complete for any t , see [144]), to perform numerical computations.

The ideas developed here could also be applied to relativistic systems, particularly to the free particle in de-Sitter space-time, where the ordinary formulation of quantum theory does not find a natural physical vacuum [148]. In this sense, the generalization of our approach to the relativistic case would provide a hierarchy of states where the first state, the relativistic counterpart of the Gaussian wave packet, plays the role of a vacuum [149].

These states could also be of practical interest in the transmission of quantum information. As a possible application we can consider the transmission of digital information, encoded by the number of humps.

This encoding would be rather robust in the sense that the number of humps is conserved even in the presence of small perturbations. Numerical calculations in one dimension have been performed, simulating perturbations by square potentials (well or barriers), leading to the conclusion that this holds as long as the mean energy of the state is large compared with the scale of the perturbing potential and the wave packet is sharp enough in momentum space in such a way that the transmission coefficient can be considered a constant. Under these circumstances (see for instance [150]), the wave packet behaves as a plane wave and the effect of the barrier in the transmitted packet is an overall attenuation, preserving its shape, and a time delay which takes its maximum values

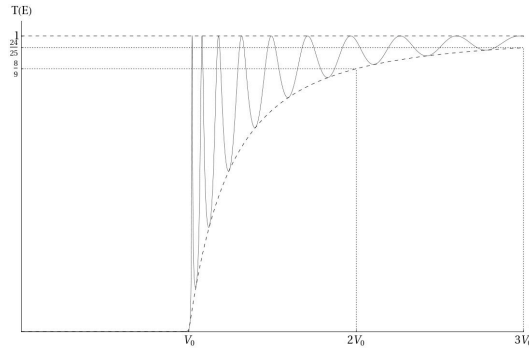


Figure 6.4: Transmission coefficient for the square barrier.

for energies near the resonant ones (and where the transmission coefficient is one). As shown in [150], this result is valid for any bounded potential of compact support, provided that the width of the potential is small in the sense that the time to pass through the barrier is smaller than the dispersion time of the wave packet τ . Therefore, the conclusions obtained with the square potential can be generalized to any finite-range bounded potential.

In Figure 6.4, the transmission coefficient $T(E)$ for a square barrier as a function of the energy E of the incident plane wave is shown. The values of $T(E)$ for values of $E = 2V_0$ and $E = 3V_0$ have been singularized, where V_0 is the height of the barrier. For $E > 2V_0$, $\frac{8}{9} < T(E) \leq 1$, and for $E > 3V_0$, $\frac{24}{25} < T(E) \leq 1$. Therefore, if the wave packet has mean energy high enough, it penetrates the barrier without distortion and practically without attenuation, with only a time delay which can be appreciable for the resonant energies $E = V_0 + \frac{\hbar^2 \pi^2}{2mb^2} n^2$, where $n = 1, 2, \dots$, and b is the width of the barrier.

It should be stressed that, for $E < V_0$ the reflection is practically total (no transmission), and that for $E \approx V_0$, $T(E)$ varies very rapidly. Thus, the wave packet should be extremely narrow in momentum to avoid distortion. However, for $E > 2V_0$ it is enough to have $\Delta p \leq \frac{\hbar \pi}{2b}$ (half the period of the oscillations of $T(E)$).

These wave packets are also robust under the influence of time dependent, homogeneous external fields, or even linear damping. In these cases, the centroid of the wave packets follow the classical trajectories, but their shapes are unaltered, apart from the unavoidable dispersion [141]. The case of linear damping is interesting due to the fact that the presence of damping prevents the dispersion of the waves packets, which asymptotically have finite width [118, 141].

6.2.5 Experimental realization

The preparation of this kind of discretized free states might be achieved by the use of a harmonic oscillator the potential of which is switched off at a given time. The vacuum state of this harmonic oscillator, when switched off, will provide the “vacuum” Gaussian wave packet with width $L = \sqrt{\frac{\hbar}{2m\omega}}$, where m is the mass of the particle and ω the frequency of the oscillator. Note that the dispersion time τ coincides with the inverse of the frequency of the oscillator. If the harmonic oscillator is in the n -th excited state, the $(n + 1)$ -hump state is obtained. To obtain traveling states, the initial state should be

a coherent state $\phi_a(x, t)$ of eq. (6.106) for a one-hump traveling state or $\phi_a^n(x, t)$ for a $(n + 1)$ -hump traveling state. These coherent states can be obtained by acting with time-dependent classical forces on the harmonic oscillator according to Glauber [147, 144]. In fact, if the classical force is given by the potential $V(x) = -f(t)x$, and the initial state is the vacuum $|0\rangle$, then a standard coherent state $|a\rangle$ is obtained with $a = \frac{i}{\sqrt{2}\omega} \hat{f}(\omega)$, where $\hat{f}(\omega)$ is the Fourier component of $f(t)$ in the frequency ω of the oscillator. This can be deduced from the study of a non-homogeneous LODE.

To avoid the dispersion effect, the traveling time of these wave packets should be less than the dispersion time τ . This would seem a severe limit for the distances that the packets can travel being localized, but this is not the case. For instance, a proton with velocity 10^4 m/s with $\Delta x = 0.1 \text{ mm}$ can travel a distance of 10^3 m while keeping localized (it is even longer for a heavier ion under the same circumstances), and this is more than enough for practical applications in Quantum Information theory.

Under the conditions commented in the previous section, these wave packets evolve without distortion even in the presence of perturbations. However, one could be interested, acting with appropriate potentials, in obtaining transitions between wave packets with different number of humps, in such a way that, for instance, a one-hump packet splits into a two-hump packet or a two-hump packet coalesces into a one-hump packet. This would open the door to performing quantum gates acting on q-bits realized with the one-hump and the two-hump states. A way of implementing this is to benefit from the fact that wave packets dynamics is similar to wave optics in the sense that an analogous to the ABCD law for optics is satisfied for wave packets [151]. Even the transmutation of Hermite-Gauss wave packets into Laguerre-Gauss wave packets can be achieved using ABCD matrices, implementing a mode converter [152].

Finally, to detect this states and measure the number of humps, the number operator \hat{N} could be used since its expectation value is directly related to the number of humps, see (6.111), once the initial position x_0 and the mean velocity v_0 are known.

It should be stressed that these states are physically observable and measurable. Let us consider, for instance, a two-hump wave packet $\phi_{\vec{a}}^{(1,0,\dots,0)}(\vec{x}, t)$ in two or three dimensions with the humps in the transversal direction to that of the mean velocity \vec{v}_0 . The separation of the two maxima of $|\phi_{\vec{a}}^{(1,0,\dots,0)}|^2$ (see Fig. 6.1) is greater, in a factor 1.6, than the uncertainty in position Δx_1 . Therefore the two humps should be measurable, and in fact, if this wave packet propagates in a bubble or wire chamber, two parallel, divergent tracks would be observed (if times $t \ll \tau$ are considered). For a three-hump wave packet, the separation among consecutive maxima (see Fig. 6.1) is smaller than the uncertainty in position, although the distance between the more separated maxima is greater than the uncertainty in position. This, together with the fact that the central maximum is smaller than the external ones, suggests that only two, overlapping thick tracks would be observed in a bubble or wire chamber. A similar behavior for a larger number of humps is expected. These behaviors are in fact observable for Hermite-Gauss states in lasers [142].

6.3 Deriving dissipative forces from a symmetry

Even though it is possible to set up a clear framework to deal with any LODE-type quantum system by employing the quantum Arnold transformation, it does not provide

by itself a well-defined operator associated with proper time evolution. This is rooted in the fact that this conventional time evolution is not included in the symmetry group that can be imported from the free system: the Hamiltonian does not belong to the specific representation of the Schrödinger algebra. One may wonder what happens if time evolution symmetry is forced. We shall pursue this issue for the damped harmonic oscillator in this Section.

6.3.1 Time symmetry

Historically, Caldirola and Kanai derived their Hamiltonian from the Bateman one by means of time-dependent canonical transformations. Now we are going to proceed in the opposite direction, deriving Bateman Hamiltonian from Caldirola-Kanai one by purely symmetry considerations.

In the damped harmonic oscillator, the operator $i\hbar\frac{\partial}{\partial t}$, nor $\hat{H}_{DHO}(t)$ close under commutation with $\hat{x}(t)$ and $\hat{p}(t)$. We wonder if it is possible to incorporate them into the basic Lie algebra of operators, trying to close an enlarged Lie algebra acting on the (possibly enlarged) Hilbert space \mathcal{H}_t . The answer to this question is in the affirmative, but it requires a delicate analysis. The resulting enlarged algebra includes $\hat{X} \equiv \hat{x}(t)$, $\hat{P} \equiv \hat{p}(t)$, $\hat{H} \equiv i\hbar\frac{\partial}{\partial t}$ and four more generators (plus \hat{I}), denoted by \hat{Q} , $\hat{\Pi}$, \hat{G}_1 and \hat{G}_2 .

The operators \hat{Q} and $\hat{\Pi}$ (plus \hat{I}) expand a Heisenberg algebra, and \hat{H} , \hat{G}_1 and \hat{G}_2 expand a 2-D affine algebra (with \hat{H} acting as dilations). However, in this realization \hat{Q} and $\hat{\Pi}$ are not basic (this can be seen as an anomaly), and \hat{H} and \hat{G}_1 are basic, resulting in time being a canonical variable. Clearly, this is not satisfactory, and an alternative description should be looked for.

A detailed study of the (projective) representations of the enlarged (7+1) dimensional Lie algebra shows that there are three relevant kinds of representations, describing systems with different degrees of freedom:

- A generic family with 3 degrees of freedom: (\hat{X}, \hat{P}) , $(\hat{Q}, \hat{\Pi})$ and (\hat{H}, \hat{G}_1) , time being a canonical variable.
- An anomalous family with 2 degrees of freedom: (\hat{X}, \hat{P}) and (\hat{H}, \hat{G}_1) , time being a canonical variable (the one already described).
- A family with 2 degrees of freedom: (\hat{X}, \hat{P}) and $(\hat{Q}, \hat{\Pi})$.

Clearly, the interesting case is the third one, since it contains two degrees of freedom

and time is not a canonical variable. Its algebra is given by:

$$\begin{aligned}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} & [\hat{Q}, \hat{\Pi}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= -\frac{i\hbar}{m} \hat{G}_2 & [\hat{X}, \hat{\Pi}] &= i\hbar \hat{G}_1 \\
[\hat{Q}, \hat{P}] &= i\hbar \hat{G}_1 - i\hbar \gamma \hat{G}_2 & [\hat{P}, \hat{\Pi}] &= -i\hbar m \omega^2 \hat{G}_2 \\
[\hat{H}, \hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} & [\hat{H}, \hat{P}] &= i\hbar m \omega^2 \hat{Q} \\
[\hat{H}, \hat{Q}] &= \frac{i\hbar}{m} (\hat{P} + 2\hat{\Pi}) & [\hat{H}, \hat{\Pi}] &= i\hbar m \omega^2 (\hat{X} - 2\hat{Q}) \\
&+ i\hbar \gamma \hat{Q} & &- i\hbar \gamma \hat{\Pi} \\
[\hat{H}, \hat{G}_1] &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2 & [\hat{H}, \hat{G}_2] &= -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2. \tag{6.119}
\end{aligned}$$

In this case the operators \hat{G}_1 and \hat{G}_2 are gauge, and therefore are represented trivially. The effective dimension of the algebra is 5 + 1: $(\hat{X}, \hat{P}), (\hat{Q}, \hat{\Pi}), \hat{H}$ and \hat{I} .

$$\begin{aligned}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} & [\hat{Q}, \hat{\Pi}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= 0 & [\hat{X}, \hat{\Pi}] &= 0 \\
[\hat{Q}, \hat{P}] &= 0 & [\hat{P}, \hat{\Pi}] &= 0 \\
[\hat{H}, \hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} & [\hat{H}, \hat{P}] &= i\hbar m \omega^2 \hat{Q} \\
[\hat{H}, \hat{Q}] &= \frac{i\hbar}{m} (\hat{P} + 2\hat{\Pi}) & [\hat{H}, \hat{\Pi}] &= i\hbar m \omega^2 (\hat{X} - 2\hat{Q}) \\
&+ i\hbar \gamma \hat{Q} & &- i\hbar \gamma \hat{\Pi}. \tag{6.120}
\end{aligned}$$

Here \hat{H} is not a basic operator, and can be written in terms of the basic ones in an irreducible representation:

$$\hat{H} = -\frac{1}{m} \hat{\Pi} \hat{P} - \frac{\gamma}{2} (\hat{Q} \hat{\Pi} + \hat{\Pi} \hat{Q}) - \frac{\hat{\Pi}^2}{m} + m \omega^2 \hat{X} \hat{Q} - m \omega^2 \hat{Q}^2. \tag{6.121}$$

The classical version of the Hamiltonian is:

$$H = -\frac{1}{m} \Pi P - \gamma Q \Pi - \frac{\Pi^2}{m} + m \omega^2 X Q - m \omega^2 Q^2. \tag{6.122}$$

6.3.2 Bateman's system

This classical Hamiltonian can be transformed, using the linear, constant, canonical transformation:

$$\begin{aligned}
 X &= \frac{m\omega^2 y - (p_y + m\frac{\gamma}{2}x)i\Omega}{m\omega\sqrt{-\gamma i\Omega}} \\
 P &= \frac{\omega(p_x - m\frac{\gamma}{2}y + mx i\Omega)}{\sqrt{-\gamma i\Omega}} \\
 Q &= \frac{m\omega^2 y - (p_y - m\frac{\gamma}{2}x)i\Omega}{m\omega\sqrt{-\gamma i\Omega}} \\
 \Pi &= -\frac{\omega(p_x + m\frac{\gamma}{2}y + mx i\Omega)}{\sqrt{-\gamma i\Omega}}, \tag{6.123}
 \end{aligned}$$

into the Bateman dual Hamiltonian

$$H_B = \frac{p_x p_y}{m} + \frac{\gamma}{2}(y p_y - x p_x) + m\Omega^2 x y, \tag{6.124}$$

that describes a damped particle (x, p_x) and its time reversal (y, p_y) :

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0, \quad \ddot{y} - \gamma\dot{y} + \omega^2 y = 0. \tag{6.125}$$

The quantum Bateman Hamiltonian is:

$$\hat{H}_B = \frac{\hat{p}_x \hat{p}_y}{m} + \frac{\gamma}{2}(\hat{y} \hat{p}_y - \hat{x} \hat{p}_x) + m\Omega^2 \hat{x} \hat{y}, \tag{6.126}$$

and the Schrödinger equation for the Bateman's system is given by¹⁴:

$$i\hbar \frac{\partial \phi(x, y, t)}{\partial t} = \left[-\frac{\hbar^2}{m} \frac{\partial^2}{\partial x \partial y} - i\hbar \frac{\gamma}{2} (y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x}) + m\Omega^2 x y \right] \phi(x, y, t). \tag{6.127}$$

The system is conservative, so our objective of including time evolution among the symmetries has been accomplished. \hat{H}_B closes a 5+1 dimensional algebra with (\hat{x}, \hat{p}_x)

¹⁴The Bateman system admits an equivalent description in terms of a real, first-order, Schrödinger equation (see [131])

and (\hat{y}, \hat{p}_y) :

$$\begin{aligned}
[\hat{x}, \hat{p}_x] &= i\hbar \hat{I} & [\hat{y}, \hat{p}_y] &= i\hbar \hat{I} \\
[\hat{x}, \hat{y}] &= 0 & [\hat{x}, \hat{p}_y] &= 0 \\
[\hat{y}, \hat{p}_x] &= 0 & [\hat{p}_x, \hat{p}_y] &= 0 \\
[\hat{H}_B, \hat{x}] &= \frac{i\hbar}{m}(-\hat{p}_y + m\frac{\gamma}{2}x) & [\hat{H}_B, \hat{p}_x] &= i\hbar(-\frac{\gamma}{2}\hat{p}_x + m\Omega^2\hat{y}) \\
[\hat{H}_B, \hat{y}] &= \frac{i\hbar}{m}(-\hat{p}_x - m\frac{\gamma}{2}y) & [\hat{H}_B, \hat{p}_y] &= i\hbar(\frac{\gamma}{2}\hat{p}_y + m\Omega^2\hat{x}). \tag{6.128}
\end{aligned}$$

However, it has been argued that the quantum Bateman's system possesses inconsistencies, like complex eigenvalues and non-normalizable eigenstates. But Chruściński & Jurkowski [169] showed that \hat{H}_B has real, continuous spectrum, and that the complex eigenvalues are associated with resonances, which in last instance are the responsible of dissipation.

6.3.3 Bateman's group law

The Lie algebra (6.128) can be exponentiated to give a Lie group, whose group law we have found to be:

$$t'' = t' + t \tag{6.129}$$

$$x'' = x + x' e^{-\frac{\gamma t}{2}} \cos \Omega t + \frac{p'_y}{m\Omega} e^{-\frac{\gamma t}{2}} \sin \Omega t \tag{6.130}$$

$$y'' = y + y' e^{\frac{\gamma t}{2}} \cos \Omega t + \frac{p'_x}{m\Omega} e^{\frac{\gamma t}{2}} \sin \Omega t \tag{6.131}$$

$$p''_x = p_x + p'_x e^{\frac{\gamma t}{2}} \cos \Omega t - m\Omega y' e^{\frac{\gamma t}{2}} \sin \Omega t \tag{6.132}$$

$$p''_y = p_y + p'_y e^{-\frac{\gamma t}{2}} \cos \Omega t - m\Omega x' e^{-\frac{\gamma t}{2}} \sin \Omega t \tag{6.133}$$

$$\zeta'' = \zeta' \zeta e^{\frac{i}{\hbar} \{ y' p_y e^{\frac{\gamma t}{2}} \cos \Omega t - x' p'_x e^{\frac{\gamma t}{2}} \cos \Omega t + m\Omega x' y' e^{\frac{\gamma t}{2}} \sin \Omega t + \frac{1}{m\Omega} p'_x p_y e^{\frac{\gamma t}{2}} \sin \Omega t \}}. \tag{6.134}$$

This group law had not been considered previously in the literature, up to the author's knowledge.

The corresponding left-invariant vector fields can be computed:

$$\begin{aligned}\tilde{X}_t^L &= \frac{\partial}{\partial t} + \left(-\frac{\gamma}{2}x + \frac{p_y}{m}\right)\frac{\partial}{\partial x} + \left(\frac{\gamma}{2}y + \frac{p_x}{m}\right)\frac{\partial}{\partial y} \\ &\quad + \left(\frac{\gamma}{2}p_x - m\Omega^2 y\right)\frac{\partial}{\partial p_x} + \left(-\frac{\gamma}{2}p_y - m\Omega^2 x\right)\frac{\partial}{\partial p_y}\end{aligned}\quad (6.135)$$

$$\tilde{X}_x^L = \frac{\partial}{\partial x} - \frac{p_x}{\hbar}\Xi \quad (6.136)$$

$$\tilde{X}_y^L = \frac{\partial}{\partial y} \quad (6.137)$$

$$\tilde{X}_{p_x}^L = \frac{\partial}{\partial p_x} \quad (6.138)$$

$$\tilde{X}_{p_y}^L = \frac{\partial}{\partial x} + \frac{y}{\hbar}\Xi, \quad (6.139)$$

and also the right-invariant ones:

$$\tilde{X}_t^R = \frac{\partial}{\partial t} \quad (6.140)$$

$$\tilde{X}_x^R = e^{-\frac{\gamma t}{2}} \cos\Omega t \frac{\partial}{\partial x} - m\Omega e^{-\frac{\gamma t}{2}} \sin\Omega t \frac{\partial}{\partial p_y} \quad (6.141)$$

$$\tilde{X}_y^R = e^{\frac{\gamma t}{2}} \cos\Omega t \frac{\partial}{\partial y} - m\Omega e^{\frac{\gamma t}{2}} \sin\Omega t \frac{\partial}{\partial p_x} + \frac{1}{\hbar}(p_y e^{\frac{\gamma t}{2}} \cos\Omega t + m\Omega x e^{\frac{\gamma t}{2}} \sin\Omega t)\Xi \quad (6.142)$$

$$\tilde{X}_{p_x}^R = e^{\frac{\gamma t}{2}} \cos\Omega t \frac{\partial}{\partial p_x} + \frac{1}{m\Omega} e^{\frac{\gamma t}{2}} \sin\Omega t \frac{\partial}{\partial y} - \frac{1}{\hbar}(x e^{\frac{\gamma t}{2}} \cos\Omega t - \frac{1}{m\Omega} p_y e^{\frac{\gamma t}{2}} \sin\Omega t)\Xi \quad (6.143)$$

$$\tilde{X}_{p_y}^R = e^{-\frac{\gamma t}{2}} \cos\Omega t \frac{\partial}{\partial p_y} + \frac{1}{m\Omega} e^{-\frac{\gamma t}{2}} \sin\Omega t \frac{\partial}{\partial x}. \quad (6.144)$$

These vector fields close the Lie algebra (6.128), provided that obvious identifications are made.

6.4 A note on the quantization of the Bateman's dual system

6.4.1 First-order Schrödinger equation

Usual Canonical Quantization leads to either position space or momentum space representation and a corresponding second-order Schrödinger equation. However, inspecting the Bateman's Lie algebra, it is possible to check that a full first-order polarization exists:

$$\mathcal{D} = \langle \tilde{X}_y^L, \tilde{X}_{p_x}^L, \tilde{X}_t^L \rangle.$$

The first two polarization conditions determine that wave functions are ($U(1)$ -functions) depending only on (x, p_y, t) . The last polarization equation $\tilde{X}_t^L \psi = 0$ determines the condition on functions on the reduced space, $\phi(x, p_y, t)$:

$$\frac{\partial \phi}{\partial t} = -\left(-\frac{\gamma}{2}x + \frac{p_y}{m}\right) \frac{\partial \phi}{\partial x} - \left(-\frac{\gamma}{2}p_y - m\Omega^2 x\right) \frac{\partial \phi}{\partial p_y}.$$

We have arrived at a first-order partial differential equation that must be interpreted as a first-order Schrödinger equation in a mixed representation position-momentum. In fact, the same result can be obtained performing Canonical Quantization for the Bateman Hamiltonian \hat{H}_B in this mixed representation. Let us emphasize that this has been strongly suggested by the group structure and the GAQ algorithm.

The corresponding time-independent Schrödinger equation is written:

$$\left(\frac{\gamma}{2}x - \frac{p_y}{m}\right) \frac{\partial \phi}{\partial x} + \left(\frac{\gamma}{2}p_y + m\Omega^2 x\right) \frac{\partial \phi}{\partial p_y} = E\phi.$$

The general solution of this equation can be found in terms of the complex variable $z \equiv p_y + im\Omega x$:

$$\phi(z) = \left(\frac{z}{z^*}\right)^{\frac{E}{2\hbar\Omega}} f\left(zz^* \left(\frac{z}{z^*}\right)^{\frac{i\gamma}{2\Omega}}\right),$$

where f is an arbitrary function of its argument.

Let us focus in the case of underdamping, where Ω is real. We must determine whether ϕ is well defined. To this end, we extract from f a power of its argument, $(zz^*)^{\tilde{\lambda}} \left(\frac{z}{z^*}\right)^{\frac{i\gamma\tilde{\lambda}}{2\Omega}}$, so that we are left with another arbitrary function g of the argument. We write:

$$\phi(z) = \left(\frac{z}{z^*}\right)^{\frac{E}{2\hbar\Omega}} (zz^*)^{\tilde{\lambda}} \left(\frac{z}{z^*}\right)^{\frac{i\gamma\tilde{\lambda}}{2\Omega}} g\left(zz^* \left(\frac{z}{z^*}\right)^{\frac{i\gamma}{2\Omega}}\right) = \left(\frac{z}{z^*}\right)^{\frac{E+i\hbar\tilde{\lambda}\gamma}{2\hbar\Omega}} (zz^*)^{\tilde{\lambda}} g\left(zz^* \left(\frac{z}{z^*}\right)^{\frac{i\gamma}{2\Omega}}\right).$$

Now, the function g and the rest of the wave function have to be well-defined independently. This restricts the possible g 's. But it also imposes a "quantization" condition on the spectrum.

On the one hand, recall that zz^* is real. For ϕ to be at least Dirac-delta normalizable, $\tilde{\lambda}$ must be chosen to be pure imaginary:

$$\tilde{\lambda} = i\lambda, \quad \lambda \in \mathbb{R}.$$

On the other hand, $\frac{z}{z^*}$ is a pure phase, with twice the argument of z . The exponent of $\frac{z}{z^*}$ must be half-integer so that we can get a well-defined function of z :

$$E - \hbar\gamma\lambda = n\hbar\Omega \quad \Rightarrow \quad \boxed{E = n\hbar\Omega + \lambda\hbar\gamma}.$$

That is, we obtain a spectrum which has an integer part and a continuous part.

These results coincide with those in [169], although they are obtained in a quicker and neater way. The reason is that they quantize angular variables and hence the basic operator “multiply by the angle” is not defined. We have avoided this problem. However, it is somewhat surprising that both spectrums coincide.

6.4.2 Back to Caldirola-Kanai system

Historically, Bateman firstly derived H_B , and later Caldirola and Kanai obtained H_{DHO} using time-dependent canonical transformations. Here we have gone the opposite way, started from H_{DHO} and derived H_B closing a finite Lie algebra. Now we wonder if we can do the way back to the Caldirola-Kanai system. The answer, again, is positive, and can be achieved by using constraints. To know how to proceed, let us analyse first the classical case.

Classically, Bateman’s system and a pair of dual Caldirola-Kanai systems share the same second-order equations of motion. If we impose them to share the first-order, Hamilton equations, the following constraint must be satisfied:

$$\begin{aligned} y &= \frac{\omega^2}{\Omega^2} e^{\gamma t} x + \frac{\gamma}{2m\Omega^2} p_x \\ p_y &= e^{\gamma t} p_x + m \frac{\gamma}{2} x. \end{aligned} \tag{6.145}$$

These constraints, although time dependent, preserve the equations of motion since they are equivalent to a relation among initials constants:

$$\begin{aligned} y_0 &= \frac{\omega^2}{\Omega^2} x_0 + \frac{\gamma}{2m\Omega^2} p_{x0} \\ p_{y0} &= p_{x0} + m \frac{\gamma}{2} x_0. \end{aligned} \tag{6.146}$$

These constraints can be seen to be of second-order type, besides being time-dependent, therefore care should be taken when imposing them: Dirac theory for constraints can be used or we can embed the constraints in a time-dependent canonical transformation before applying them.

But we are interested in the quantum derivation. Therefore we try to impose the operator constraints:

$$\begin{aligned}\hat{y} - \frac{\omega^2}{\Omega^2} \hat{x} - \frac{\gamma}{2m\Omega^2} \hat{p}_x &= 0 \\ \hat{p}_y - \hat{p}_x - m \frac{\gamma}{2} \hat{x} &= 0,\end{aligned}\tag{6.147}$$

but only one of them can be imposed, since the operators at the lhs of the equations canonically commute: they are of second order type. At the quantum level, only one of them can be imposed, therefore we must select one of them. If we impose the constraint,

$$\hat{y} = \frac{\omega^2}{\Omega^2} \hat{x} + \frac{\gamma}{2m\Omega^2} \hat{p}_x,\tag{6.148}$$

the Hilbert space reduces to those functions verifying:

$$\phi(x, y, t) = e^{\frac{ie^{-\gamma t} m \Omega y \operatorname{Csc}^2(\Omega t) (\gamma \Omega y \operatorname{Cos}(2\Omega t) + 2(\omega^2 e^{\gamma t} x' - \Omega^2 y) \operatorname{Sin}(2\Omega t))}{4\hbar \omega^2}} \psi(x', t),\tag{6.149}$$

where $x' = x + \frac{\Omega^2}{2\omega^2} y e^{-\gamma t} \mu(t)$, and $\mu(t) = (2 - \frac{\gamma}{\Omega} \operatorname{Cot}(\Omega t))$. The Schrödinger equation for the Bateman's system reduces to:

$$i\hbar \frac{\partial \psi(x', t)}{\partial t} = \left[-\frac{\Omega^2 \hbar^2}{2m\omega^2} e^{-\gamma t} \mu(t) \frac{\partial^2}{\partial x'^2} - \frac{1}{2} i\hbar x' \Omega \mu(t) \frac{\partial}{\partial x'} + i\hbar \frac{\Omega^2}{\gamma} (\mu(t) - 2) \right] \psi(x', t).\tag{6.150}$$

When constraints are imposed, not all the operators acting on the original Hilbert space preserve the constrained Hilbert space. The notion of “good” (usually denoted gauge-independent in constrained gauge theories) operators as those preserving the constrained Hilbert space naturally emerges.

In most of the cases “good” operators are characterized as those commuting with the constraints (see [18] for a detailed account of quantum constraints in a group-theoretical setting and a more general characterization of “good” operators). In this case they are:

$$\hat{p}_x + \frac{2m\omega^2}{\gamma} \hat{x} \quad \hat{p}_y - \frac{2m\Omega^2}{\gamma} \hat{x}.\tag{6.151}$$

Note that \hat{H}_B (nor $i\hbar \frac{\partial}{\partial t}$) is not among the “good” operators since it does not preserve the constrained Hilbert space. Therefore, time invariance is lost in the process of going from the Bateman's system to the Caldirola-Kanai system due to the very nature of the constraints imposed.

Now let us perform the transformation

$$\psi(x', t) = e^{-i \frac{m\omega^2}{\hbar\Omega} x'^2 f(t)} g(t) \chi(\kappa, \tau),\tag{6.152}$$

where

$$f(t) = -\frac{e^{\gamma t}}{4\Omega\mu(t)^2\tau'(t)} \left((-\gamma(2 + \text{Cos}(2\Omega t)) + 2\Omega\text{Sin}(2\Omega t))\tau'(t) - \gamma\mu(t)\tau'(t)^2 + \mu(t)\tau''(t) \right) \quad (6.153)$$

$$g(t) = e^{-\frac{1}{4}\gamma\tau} \left(-\frac{\tau'(t)}{\Omega\text{Sin}^2(\Omega t)\mu(t)} \right)^{1/4} \quad (6.154)$$

$$\kappa = x' e^{\frac{\gamma}{2}(t-\tau)} \frac{\omega}{\Omega} \sqrt{\frac{\tau'(t)}{\mu(t)}} \quad (6.155)$$

$$\tau(t) = \frac{1}{\Omega} \text{ArcTan} \left[\frac{A\frac{\gamma^2}{\Omega^2}}{\mu(t)^2} \right], \quad A \in \mathbb{R} - \{0\}. \quad (6.156)$$

The Schrödinger equation finally transforms into:

$$i\hbar \frac{\partial}{\partial \tau} \chi(\kappa, \tau) = \left[-\frac{\hbar^2}{2m} e^{-\gamma\tau} \frac{\partial^2}{\partial \kappa^2} + \frac{1}{2} m\omega^2 \kappa^2 e^{\gamma\tau} \right] \chi(\kappa, \tau), \quad (6.157)$$

which is the Caldirola-Kanai equation in the variables (κ, τ) . Even more, the two independent operators (6.151) preserving the constrained Hilbert space turn, under the previous transformation, to the basic operators for the Caldirola-Kanai system $\hat{x}(t)$ and $\hat{p}(t)$. Therefore, we have recovered completely the Caldirola-Kanai system from the Bateman's system by imposing one constraint.

It should be stressed that $\tau'(0) = 0$, therefore the time transformation is singular at the origin and there are two disconnected regions, one with $t > 0$ and other with $t < 0$. It also turns out that $\text{sign}(\tau) = \text{sign}(A)$, therefore choosing appropriately the sign of A in each case we can map $t > 0$ to $\tau > 0$ and $t < 0$ to $\tau < 0$, respectively.

This kind of behavior coincides with the results of other authors (see [169]) where, starting with the Bateman's system, they obtain two subspaces \mathcal{S}^\pm for which the restriction of the one parameter group of unitary time-evolution operators $\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}_B}$ produces two semigroups of operators, for $t < 0$ and $t > 0$.

Therefore, starting from the quantum, conservative, Bateman's system we have arrived to the quantum, time-dependent, Caldirola-Kanai system. All the process we have performed can be schematically showed as:

$$\begin{array}{ccc}
 & \text{Constraint} & \\
 & \Rightarrow & \\
 \text{Bateman} & & \text{Caldirola-Kanai} \\
 t \in \mathbb{R} & & t \in \mathbb{R}^+ \text{ or } t \in \mathbb{R}^- \\
 \text{Conservative} & & \text{Dissipative} \\
 & \text{Closing algebra} & \\
 & \Leftarrow &
 \end{array} \quad (6.158)$$

Chapter 7

Conclusions

The main results of this thesis could be summarized as follows:

1. We have provided an algorithm to identify functions of the basic coordinates of the solution manifold which close a Lie algebra for a number of mechanical systems, such as a particle moving in the $SU(2)$ manifold (non-linear sigma model for $SU(2)$), the Poschl-Teller potential or the Morse potential. The quantum theory of these systems can be studied then by the Group Approach to Quantization method and it has been illustrated in the case of the sigma model.
2. The quantum theory of free particle in the sphere \mathbb{S}^2 , as a non-linear sigma model on $SU(2)/U(1)$ has been studied. The quantum symmetry of the system has been identified. As a remarkable advantage over other quantization methods, GAQ is coordinate-free, in such a way that the quantum operators are automatically selected. In particular, this implies that there is no need to impose any constraint.

We would like to point out that a similar strategy can be explicitly adopted in dealing with particles moving on coadjoint orbits of semi-simple groups, non-necessarily spheres, as shown in our general scheme.

In dealing with less symmetrical system, yet keeping topologically equivalent solution manifold, the present approach could be in principle generalized by making use of geometrically different realizations of the same algebraic group structure. These physically inequivalent realizations of the same group would be associated with diffeomorphisms of the symplectic solution manifold which are not contact transformations [76]. A similar idea can be found in [77] (Chap. 10), where non-holonomic mappings are used to relate non-equivalent physical systems.

3. We have provided a consistent quantization of the Poisson algebra among the basic functions on the solution manifold of massive Yang-Mills fields coupled to non-linear partial-trace sigma scalar fields. We have looked for unitary and irreducible representations of the group named G_{MYM} in the text. As already commented, the parameters of this group have been written as functions on the Cauchy surface Σ . The Poincaré subgroup has been omitted but we have kept manifest covariance thoroughly. In order to recover the time evolution at the quantum level we have constructed a Hamiltonian operator \hat{H} , uniquely defined in terms of the basic operators, which preserves the quantum representation space, the Hilbert space. This

allows, in principle, any proper computation concerning the time evolution. For instance, one can derive a perturbation theory à la Heisenberg [171], that is, using the complete Hamiltonian, and evolving the wave functions from the solution manifold. A precise way of so doing is by making use of the Magnus expansion [136], which preserves unitarity at each order in the exponential of the Hamiltonian, as a generator of time translation. The actual computation of this series and the connection with the more traditional expansion of the S matrix would be in order. But an alternative though equivalent way of approaching a perturbative expansion of the S matrix can also be realized in a completely algebraic manner. In fact, starting from our algebra of basic operators (4.45), we may proceed by closing a new Lie algebra by commutation with \hat{H} , order by order in a formal “expansion” constant α , exponentiating the resulting algebra up to the same order in α , and requantizing again with the present group-theoretical method. Much work is being made in this direction.

In either perturbative scheme we hope to achieve, in particular, an approach to the quantum description of the Physics around the Particle Standard Model which is the final target of our present algorithm.

4. We have considered a non-canonical approach to the perturbation theory of the $O(N)$ -invariant NLSM which accounts for the non-trivial (non-flat) geometry and topology of the target manifold Σ and takes advantage of the underlying symmetries of the system. This scheme can also be adapted to other G -invariant NLSM. The usual perturbation theory for relativistic fields is designed for small deviations from the free (Klein-Gordon or Dirac) fields. Detectors in large particle colliders are also designed for this purpose. However, fields of NLSM-type can be found in a strongly-interacting regime ($\rho \ll 1$) which does not fit into this picture. This should lead to reconsider the perturbation theory and renormalizability of the NLSM.
5. The analysis carried out with the Quantum Arnold Transformation permits to deal with the quantum theory of any LSODE-type dynamical system, using known properties of the quantum free particle. The QAT provides basic operators and establishes that the symmetry group of the free particle, the Schrödinger group, can be transferred to a realization on the LSODE system. This result turns out to be of practical use when performing some computations, for instance finding solutions of the Schrödinger equation or the evolution operator, especially when the Hamiltonian does not commute with itself at different times. Even in these cases it is possible to give exact expressions, obtained in a non-perturbative manner. It is noteworthy that these calculations lead to the knowledge of objects in the quantum theory with the only requirement that the classical solutions of the LSODE are known.

In a way, Arnold transformation allows to interpret LSODE-type forces, including dissipation linear in velocity, as effects observed in a “non-inertial reference frame”. The use of the present scheme goes beyond the study of the simple damped harmonic oscillator, finding applicability in quite different branches of physics, such as Cosmology, where a scalar field appears (inflaton) satisfying equations in time which can be read as a LSODE. In this respect the QAT can be applied to the study

of the specific example of the harmonic oscillator with time-dependent frequency (see [160] and references therein). This will be considered in the next future [139].

It is worth mentioning that the Arnold transformation used here is just a particular case of a broader class of transformations which link free particle equations to even classical non-linear equations. Research in this direction would potentially lead to extremely useful and interesting results. We believe that a good starting point for this purpose was presented in [129], although some effort to establish the explicit connection with the free particle would be in order.

6. For the Caldirola-Kanai system, describing a quantum damped harmonic oscillator, a couple of constant-of-motion operators generating the Heisenberg algebra can be found using the QAT. The inclusion of the standard time evolution symmetry in this algebra for damped systems, in a unitary manner, requires a non-trivial extension of this basic algebra and hence the physical system itself. Surprisingly, this extension leads directly to the so-called Bateman's dual system, which now includes a new particle acting as an energy reservoir. The group of symmetries of the dual system has been presented, as well as a quantization that implies, in particular, a first-order Schrödinger equation. The usual second-order equation and the inclusion of the original Caldirola-Kanai model in Bateman's system has also been discussed.

Bibliography

- [1] A. Galindo and P. Pascual, *Quantum Mechanics vol. I*, Springer-Verlag, Berlin (1990).
- [2] J. van Hove, *Ascd. R. Belg., Cl. Sci., Mem., Collect.*, **6**, t. XXVI (1951).
- [3] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge Univ. Press (1991).
- [4] M. Gotay, *Obstructions to quantization*, arXiv:math-ph/9809011v1.
- [5] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
- [6] V. Aldaya and J. Guerrero, *J. Phys. A: Math. Gen.* **38**, 6939 (2005).
- [7] V. Aldaya and J.A. de Azcárraga, *J. Math. Phys.* **23**, 1297 (1982)
- [8] J.M. Souriau, *Structures des systemes dynamiques* (Dunod, Paris, 1970).
- [9] B. Kostant, *Quantization and Unitary Representations*, in *Lecture Notes in Mathematics*, **170** (Springer, Berlin, 1976).
- [10] A.A. Kirillov, *Elements of the Theory of Representations* (Springer, Berlin, Heidelberg, NY, 1976).
- [11] N. Woodhouse, *Geometric Quantization* (Clarendon, Oxford, 1980).
- [12] J.R. Klauder, *Ann. Phys.* **188**, 120 (1988).
- [13] V. Pažma and P. Prešnajder, *Eur. J. Phys.* **10**, 35 (1989).
- [14] V. Aldaya, J. Bisquert and J. Navarro-Salas, *Phys. Lett.* **A156**, 381 (1991).
- [15] V. Aldaya, J.R. Klauder and J. Navarro-Salas, *Class. Quantum Grav.* **8**, 413 (1991).
- [16] J. Navarro-Salas, M. Navarro and V. Aldaya, *Nuc. Phys.* **B403** 291 (1993).
- [17] V. Aldaya, M. Calixto and J.M. Cervero, *Commun. Math. Phys.* **200**, 325 (1999)
- [18] V. Aldaya, M. Calixto and J. Guerrero, *Commun. Math. Phys.* **178**, 399 (1996)
- [19] M. Calixto and V. Aldaya, *J. Phys.* **A32**, 7287 (1999).
- [20] V. Bargmann, *Ann. Math.*, **59**, 1 (1954).
- [21] V.V. Kozlov, A.O. Harin: *Celestial Mech. Dynam. Astron.* **54**, 393-399 (1992)
- [22] J.J. Slawianowski: *Rep. Math. Phys.* **46**, 429 (2000)
- [23] J.M. Lévy-Leblond, in: *Group Theory and its Applications*, **2**, Ed.: E.M. LoebI (Academic Press, New York, 1971).

- [24] E.C.G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective*, J. Wiley, New York, (1974).
- [25] J. Guerrero and V. Aldaya, *J. Math. Phys.* **41**, 6747 (2000).
- [26] V. Aldaya, J. Guerrero and G. Marmo, *Int. J. Mod. Phys.* **A12**, 3 (1997).
- [27] R. Jackiw, in: *Current Algebra and Anomalies*, Eds.: S.B. Treiman, R. Jackiw, B. Zumino and E. Witten (World Scientific, Singapore, 1985).
- [28] V. Aldaya, J.L. Jaramillo and J. Guerrero, *J. Math. Phys.* **44**, 5166 (2003)
- [29] R.M. Wald, *General Relativity*, The University of Chicago Press, Chicago and London (1984).
- [30] S.J. Clark and R.W. Tucker, *Class. Quantum Grav.* **17**, 4125 (2000)
- [31] R. Maartens and B.A. Bassett, *Class. Quantum Grav.* **15**, 705 (1998)
- [32] B. Mashhoon, arXiv:gr-qc/0311030v1 8Nov.2003
- [33] E. Silvestre, *Posibles cinemáticas*, Little Thesis, Valencia University (1989).
- [34] V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *J. of Nonlinear Math. Phys.* **15**, 1 (2008).
- [35] D.J. Navarro and J. Navarro-Salas, *J. Math. Phys.* **37**, 6060 (1996)
- [36] J. Mickelsson *Current Algebras and Groups*, Plenum Press (1989)
- [37] P. Goddard and D. Olive, *Int. J. Mod. Phys.* **A1**, 303 (1986); P. Goddard, A. Kent and D. Olive, *Commun. Math. Phys.* **103**, 105 (1986)
- [38] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw-Hill (1985).
- [39] M. Navarro, V. Aldaya and M. Calixto, *J. Math. Phys.* **37**, 206 (1996); **38**, 1454 (1997).
- [40] V. Aldaya and J. de Azcárraga, *Geometric formulation of classical mechanics and field theory*, *Riv. Nuov. Cimento* **n. 10** (1980)
- [41] P. Malliavin, *Géométrie différentielle intrinsèque*, Hermann Paris (1972).
- [42] V.P. Nair, *Quantum Field Theory*, Graduate Texts in Contemporary Physics, Springer (2005).
- [43] V. Aldaya, M. Calixto, J. Guerrero and F.F. López-Ruiz, *Rep. Math. Phys.* **64**, 49-58 (2009).
- [44] V. Aldaya, M. Calixto and F.F. López-Ruiz, *Mod. Phys. Lett.* **A24** (2009) 2731-2740
- [45] M. Calixto, V. Aldaya and F.F. López-Ruiz, *J. of Nonlinear Math. Phys.* **15**, 91 (2008)
- [46] H. Ruegg and M. Ruiz-Altaba, *Int. J. Mod. Phys.* **A19**, 3265-3348 (2004).
- [47] V. Aldaya, M. Calixto and E. Sánchez-Sastre, *Mod. Phys. Lett. B* **21**, 2813 (2006).
- [48] R. Delbourgo, S. Twisk, and G. Thompson, *Int. J. Mod. Phys.* **A3**, 435-449 (1988).
- [49] M. Bando, T. Kugo and K. Yamawaki, *Nucl. Phys.* **B259**, 493 (1985); *Phys. Rep.* **164**, 217 (1988).
- [50] J.R. Forshaw, J. Papavassiliou and C. Parrinello, *Phys. Rev. D* **59**, 074008 (1999).
- [51] E. C. G. Stueckelberg, *Helv. Phys. Acta* **11**, 225-244 (1938).

- [52] J. H. Lowenstein, and B. Schroer, *Phys. Rev.* **55** D6, 1553-1571 (1972).
- [53] T. Kunimasa and T. Goto, *Prog. Theor. Phys.* **37**, 452 (1967).
- [54] K.I. Shizuya, *Nucl. Phys.* **B94**, 260 (1975).
- [55] J. M. Cornwall, D. N. Levin and Tiktopoulos, *Phys. Rev.* **D10**, 1145 (1974).
- [56] J. Kubo, *Phys. Rev. Lett.* **58**, 2000 (1987).
- [57] T. Kugo, and I. Ojima, *Prog. Theor. Phys. Suppl.* **66**,1 (1979).
- [58] N. Dragon, T. Hurt, and P. van Nieuwenhuizen, *Nucl. Phys. Proc. Suppl.* **56B**, 318 (1997).
- [59] T. Hurt, *Helv. Phys. Acta* **70**, 406 (1997).
- [60] G. Efimov, *Nucl. Phys.* **74**, 657 (1965).
- [61] A. Burnel, *Phys. Rev.* **D33**, 2981 (1986).
- [62] R. Delbourgo and G. Thompson, *Phys. Rev. Lett.* **57**, 2610 (1986).
- [63] V. Aldaya, M. Calixto and J. Guerrero, *Int. J. Theor. Phys.* **35**, 1901 (1996).
- [64] V. Aldaya and E. Sánchez-Sastre, *J. Phys.* **A39**, 1729 (2006).
- [65] A. A. Slavnov, *Theor. Math. Phys.* **10**, 201 (1972).
- [66] A. A. Slavnov and L. D. Fadeev, *Theor. Math. Phys.* **3** 18 (1971).
- [67] H. Kleinert and S. V. Shabanov: *Phys. Lett. A* **232**, 327-332 (1997).
- [68] A. Scardicchio: *Phys. Lett. A* **300**, 7-17 (2002).
- [69] C. Neves and C. Wotzasek: *J. Phys. A* **33**, 6446-6456 (2000).
- [70] V. Aldaya, E. Sánchez-Sastre and M. Calixto: *Rep. Math. Phys.* **59**, 83-109 (2007).
- [71] J.-M. Lévy-Leblond: *Galilei group and Galilean invariance*. E. LoebI, Editor, *Group Theory and Applications II*, Academic Press, New York (1972), p. 222.
- [72] E. J. Saletan, *J. Math. Phys.* **2**, 1 (1961).
- [73] J. Guerrero, J.L. Jaramillo and V. Aldaya: *J. Math. Phys.* **45**, 2051 (2004).
- [74] K. Kowalski and J. Rembieliński: *J. Phys. A* **33**, 6035-6048 (2000); K. Kowalski and J. Rembieliński, *J. Math. Phys.* **42**, 4138-4147 (2001).
- [75] A. V. Golovnev, arXiv:quant-ph/0508044v3.
- [76] Work in progress.
- [77] H. Kleinert, *Path integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets*, World Scientific (2006).
- [78] E. Wigner, *Group Theory in its Applications to the Quantum Mechanics of Atomic Spectra*, Academic Press, NY (1959)
- [79] V. Aldaya, J. Bisquert, J. Guerrero and J. Navarro-Salas, *Rep. Math. Phys.* **37**, 387 (1996)
- [80] J. Guerrero and V. Aldaya, *Mod. Phys. Lett. A* **14**, 1689 (1999)
- [81] V. Aldaya, J. Bisquert, J. Guerrero and J. Navarro-Salas, *J. Phys. A* **26**, 5375 (1993)
- [82] C. Li, *Chinese J. Phys.* **32**, 309-318 (1994).

- [83] N. Rosen and P.M. Morse, *On the vibrations of Polyatomic Molecules*, Physical Review **42**, 210 (1932)
- [84] F. Scarf, Phys. Rev. **112**, 1137 (1958)
- [85] V.V. Kozlov, A.O. Harin: Celestial Mech. Dynam. Astron. **54**, 393-399 (1992)
- [86] J.J. Slawianowski: Rep. Math. Phys. **46**, 429 (2000)
- [87] J.F. Cariñena, M.F. Rañada and M. Santander, J. Math. Phys. **46**, 052702 (2005)
- [88] J.F. Cariñena, M.F. Rañada and M. Santander, Rep. Math. Phys. **54**, 285 (2004)
- [89] M.M. Nieto and L.M. Simmons, Jr., Physical Review D **20**, 1342 (1979).
- [90] J.P. Antoine, J.P. Gazeau, P. Monceau, J.R. Klauder and K. Penson, J. Math. Phys. **42** 2349, (2001)
- [91] J.M. Arias, J. Gómez-Camacho and R. Lemus, J. Phys. A **37**, 877 (2004)
- [92] M. Berrondo, J. Recamier, Int. J. Quant. Chem. **62**, 239 (1997)
- [93] J. Guerrero and V. Aldaya, J. Phys. A **39**, L267-L276 (2006)
- [94] P.W. Higgs, Phys. Rev. Lett. **13**, 508 (1964); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. Kibble, Phys. Rev. Lett. **13**, 585 (1964).
- [95] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).
- [96] G. 't Hooft, Nucl. Phys. **33**, 173 (1971); **35**, 167 (1971).
- [97] A.C. Longhitano, Nucl. Phys. B **188**, 118 (1981).
- [98] W.A. Bardeen and K. Shizuya, Phys. Rev. **D18**, 1969 (1978).
- [99] P. Kosinski and L. Szymanowski, Phys. Rev. Lett. **58**, 2001 (1987)
- [100] S.V. Ketov, *Quantum Non-linear Sigma-Models*, Springer-Verlag Berlin Heidelberg (2000).
- [101] V. Aldaya, M. Calixto and F.F. Lopez-Ruiz, Proceedings of the 28th International Colloquium on Group Theoretical Methods in Physics, New Castle 2010, UK.
- [102] V. Aldaya, M. Calixto, J. Guerrero and F.F. Lopez-Ruiz, *Symmetries of Non-Linear Systems: Group Approach to their Quantization*, in preparation.
- [103] C.J. Isham, *Topological and global aspects of quantum theory*, in *Relativity, Groups and Topology II*, Les Houches Summer School, R. Stora and B. S. DeWitt (Eds.) (1986)
- [104] V. Aldaya, M. Calixto and F.F. Lopez-Ruiz, *Symmetry group of massive Yang-Mills theories and its quantization*, in preparation.
- [105] Under investigation
- [106] A.A. Belavin and A.M. Polyakov, JETP Lett. **22** (1975) 245-247
- [107] N. E. Ligterink, N. R. Walet and R. F. Bishop, Nuclear Physics B (Proc. Suppl.) **63A-C** (1998) 667-669
- [108] N. E. Ligterink, N. R. Walet and R. F. Bishop, Ann. Phys. **267** (1998) 97-133
- [109] V. Aldaya, J. Navarro-Salas and A. Ramirez, Commun. Math. Phys. **121** (1989) 541

- [110] N.P. Landsman and N. Linden, Nucl. Phys. **B365** (1991) 121
- [111] D. MacMullan and I. Tsutsui, Ann. Phys. **237** (1995) 269-321
- [112] L.C. Biedenharn and J.D. Louck, Angular Momentum in Quantum Physics, Addison-Wesley, Reading, MA, 1981
- [113] P. Caldirola., Nuovo Cimento **18**, 393 (1941)
- [114] E. Kanai, Prog. Theor. Phys. **3**, 440 (1948)
- [115] R. Martini and P.H.M. Kersten., J. Phys. A **16**, 455 (1983)
- [116] J.M. Cerveró and J. Villarroel, J. Phys. A **17**, 1777 (1984)
- [117] F.T. Hioe, J. Math. Phys. **15**, 445 (1974); Yuen H.P., Phys. Rev. A **13**, 2226 (1976).
- [118] V.V. Dodonov and V.I. Man'ko, Phys. Rev. A **20**, 550 (1979)
- [119] W.E. Brittin, Phys. Rev. **77**, 396 (1950)
- [120] M. Razavy, *Classical and Quantum Dissipative Systems* (Imperial College Press, 2005).
- [121] D. Schuch, Phys. Rev. A **55**, 935 (1997)
- [122] V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations* (Springer, 1998)
- [123] M. Lutzky, J. Phys. A **11**, 249 (1978)
- [124] S. Takagi, Prog. Theor. Phys. **84**, 1019 (1990)
- [125] M-C. Huang, M-C. Wu, Chin. J. Phys. **36**, 566 (1998)
- [126] H. Kanasugi and H. Okada, Prog. Theor. Phys. **93**, 949 (1995)
- [127] Dae-Yup Song, Europhys. Lett. **62**, 622 (2003)
- [128] A. Mostafazadeh, Phys. Rev. A **55**, 4084 (1997)
- [129] A. Mostafazadeh, J. Phys. A **31**, 6495 (1998)
- [130] V. Aldaya, J. Navarro-Salas, J. Bisquert and R. Loll, J. Math. Phys. **33**, 3087 (1992)
- [131] J. Guerrero, V. Aldaya, F. Cossío and F.F. López-Ruiz, *A round trip from Caldirola to Bateman* in *GROUP 28: The XXVIII ICGTMP*, Newcastle 26-30 July 2010, to appear in IoP Journal of Physics Conference Series;
V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz, *A symmetry trip from Caldirola to Bateman damped systems*, to be published.
- [132] J.M. Cerveró and J. Villarroel, J. Phys. A **17**, 2963 (1984)
- [133] J.M. Lévy-Leblond, Commun. Math. Phys. **12**, 64 (1969)
- [134] V.Aldaya and J. Guerrero Rep. Math. Phys. **47**, 213 (2001)
- [135] J. Guerrero, F.F. López-Ruiz, F. Aldaya and F. Cossío, arXiv:1010.5525
- [136] S. Blanes, F. Casas, J.A. Oteo and J. Ros, Phys. Rep. **470**, 151 (2009)
- [137] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products* (Academic, 2007)
- [138] V. Aldaya, J.A. de Azcárraga and K.B. Wolf, J. Math. Phys. **25**, 506 (1984)

- [139] V. Aldaya, F. Cossío, J. Guerrero and F.F. López-Ruiz (in preparation).
- [140] V. Aldaya, F. Cossío, J. Guerrero and F. F. López-Ruiz, *On a quantum version of the Arnold transformation*, accepted for publication in J. Phys. A; arXiv:1010.5521
- [141] M. Andrews, Am. J. Phys. **67**, 336 (1999); Am. J. Phys. **76**, 1102 (2008)
- [142] B.E.A. Saleh, M.C. Teich, *Fundamentals of Fotonics*, Wiley, New Jersey (2007)
- [143] C.J. Bordé, C. R. Acad. Sci. Paris, **2**, 509–530 (2001); Phys. Rev. **A79**, 043613 (2009)
- [144] A. Perelomov, *Generalized Coherent States and their Applications*, Springer-Verlag (1986)
- [145] M. Moshinsky and Y. Smirnov, *The Harmonic Oscillator in Modern Physics*, Harwood Academic, Amsterdam (1996)
- [146] M.V. Stoitsov and I. Zh. Petkov, Ann. Phys. (N.Y.) **184**, 121 (1988)
- [147] R.J. Glauber. *Optical coherence and the photon statistics*. In C. DeWitt, A. Blandin, and C. Cohen-Tannoudji, editors, *Quantum optics and electronics*. Gorgon and Breach (1965).
- [148] B. Allen, Phys. Rev. D **32**, 3136 (1985)
- [149] In preparation
- [150] C.L. Hammer, T.A. Weber and V.S. Zidell, Am. J. Phys. **45**, 933 (1977)
- [151] Impens, F., Borde, C.H., Phys. Rev. A **79**, 043613 (2009)
- [152] Bastiaans, M.J., and Alieva, T., J. Phys. **A 38**, L73 (2005)
- [153] Scully M.O., Zubairy M.S., *Quantum Optics* (Cambridge University Press, 1997)
- [154] Breuer H-P, Petruccione F., *The Theory of Open Quantum System* (Oxford University Press 2002)
- [155] Dittrich T. et al., *Quantum Transport and Dissipation* (Wiley-VCH 1998)
- [156] Dekker H., Phys. Rev. A **16**, 2126 (1977)
- [157] Razavy M., Z. Physik B **26**, 201 (1977)
- [158] Dekker H., Phys. Rep. **80**, 1 (1981)
- [159] C. R. Hagen, Phys. Rev. D **5**, 377 (1972); Niederer U., Helv. Phys. Acta **45**, 802 (1972); **46**, 191 (1973); **47**, 167 (1974)
- [160] J.D. Lejarreta and J.M. Cerveró, Cent. Eur. J. Phys. **6**, 671 (2008)
- [161] C-I. Um, K-H. Yeon and W.H. Kahng, J. Phys. A **20** 611 (1987)
- [162] C-I. Um, K-H. Yeon, T.F. George, Phys. Rep. **362**, 63 (2002)
- [163] A.N.W. Hone, M. Senthilvelan, J. Math. Phys. **50**, 102902 (2009)
- [164] A. Streklas, Physica A, **385**, 124 (2007)
- [165] H. Bateman, Phys. Rev. **38**, 815 (1931)
- [166] Y. Tikochinsky, J. Math. Phys. **19**, 888 (1978); H. Feshbach and Y. Tikochinsky, in: A Festschrift for I.I. Rabi, Trans. New York Ac. Sc. Ser. 2. Vol. 38 (1977) 44
- [167] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics Vol. I* (McGraw-Hill, New York, 1953), p. 298.

-
- [168] E. Celeghini, M. Rasetti, G. Vitiello, *Ann. Phys.* **215**, 156 (1992)
- [169] D. Chruściński, J. Jurkowski, *Ann. Phys.* **321**, 854 (2006)
- [170] M. Blasone, P. Jizba, *Ann. Phys.* **312**, 312 (2004)
- [171] L.D. Landau and E.M. Lifshitz, *Quantum Electrodynamics*, Course of Theoretical Physics, Vol. 4, Pergamon Press (1985).